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ERRATA.

PAGE.	LINE.	
9	13	from bottom, for "$u = z$" read $u_x = z^x$.
30	8	" " " "$\sec^2\frac{1}{2}x = 2dz$" read $\sec^2\frac{1}{2}xdx = 2dz$.
32	first	" "$\frac{du}{ax}$" read $\frac{du}{dx}$.
33	last	" "$\left(\frac{y}{4ar}\right)$" read $\left(\frac{y}{2ar}\right)$.
49	last	" "$\tan \frac{\left(\sin^{-1}\frac{\tan\theta}{\tan\psi}\right)}{\sin\theta}$" read $\tan \frac{\left(\sin^{-1}\frac{\tan\theta}{\tan\psi}\right)}{\sin\theta}$.
100	15	" "$\frac{m'}{r'}\ \frac{m''}{r''}$" read $\frac{m'}{r'} + \frac{m^2}{r^2}$.
144	15	" "$\frac{41}{7560}J^{10}$" read $\frac{41}{2560}J^{10}$.
165	9	from bottom, for "p being the parameter" read p being half the parameter.
174	2	" "$c + r_1$" read $a + r_1$.
176	6 & 7 after	$(7 - 4\sqrt{3})^{n+\frac{1}{2}}$ and $(7 - 4\sqrt{3})^{n-\frac{1}{2}}$ put -4.
176	9	" $(7 - 4\sqrt{3})^n$ put -2.
180	3	from bottom, for "$\frac{r^2}{r^2\ \ x^2}$" read $\frac{r^2}{r^2 - x^2}$.
164	5	" "$n = 799$," &c. read $n = .799$, &c.
187	6	" "$\frac{\cos^2 a}{2}$" read $\frac{\cos^2 a}{4}$.
192	14	" "$(C'L)^2$ read $(OL)^2$.
190–1		" "g" in cut read q.
205	14	" "c^2" read c^3.
205	16	" "$p\quad s = e$" read $p\quad s = c$.
205	18	" "r^2" read r^3.
206	8	" "$t\quad 2y - z$" read $t = 2y - z$.
210		" "b" in the cut read b', and for "b'" read b.

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THE ANALYST.

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INTRODUCTORY REMARKS

THE PRESENT is eminently a period of activity, both physical and mental. Science is daily developing new truths, and thereby increasing the sum of human knowledge. Guided by analysis, the mechanic is daily improving and perfecting labor-saving machinery, thereby augmenting the amount of human hapiness; and the astronomer is re-examining his conclusions, and, with the help of new and improved instruments, correcting his data, thereby perfecting our knowledge of the extent and harmony of the material universe.

As a knowledge of the laws of natural phenomena (and as a consequence the happiness and welfare of mankind) is promoted by community of mind, it is believed that by such an intercourse of thought as this journal is intended to induce, the sum of human happiness will be increased.

The editor is fully aware that no effort on his part alone can make such a publication as this is intended to be generally interesting to its readers; he only hopes for success in that respect by enlisting as contributors a majority of its readers. He therefore invites all who may feel an interest in its success to contribute to its pages their best thoughts and most valuable conclusions, embodied in brief and concise notes or essays.

As the scientific characte of the ANALYST has not been fully explained by circular, we embrace this opportunity to state that, as its title imports, it is intended to afford a medium for the presentation and analysis of any and all questions of interest or importance in pure or applied Mathematics, embracing especially all new and interesting discoveries in theoretical and practical astronomy, mechanical philosophy and engineering.

We are fully aware of the difficulty of publishing such a periodical as we have above indicated, and of the apparent presumption of attempting it at this place, where we have no prominent institution of learning, nor the facilities for printing that might be obtained farther east. Nevertheless, as there seems to be an obvious want of a suitable medium of communication between a large class of investigators and students in science, comprising the various grades from the students in our high schools and colleges to the college professor; and moreover, as we have been encouraged by kind

words, and promises of assistance from various eminent teachers and professors, which, from the contributions received for this our first number, we have reason to believe will be fully realized, we have determined to venture the publication.

We invite, and expect to obtain, the following two classes of persons as readers of our Journal, viz: 1st, Those who are able and willing to communicate valuable information through the Journal; and, 2nd, Those who desire to increase their stock of knowledge and shall find that desire partly supplied by the Journrl.

All who feel an interest in the success of the Journal are respectfully solicited to co-operate with, and assist us in extending its circulation.

We earnestly solicit contributions for publication from all who desire to promote the interest and usefulness of the Journal. In selecting matter for publication each month, we will present such as we may think most interesting or of greatest utility.

We will publish from three to five mathematical questions in each number, and will endeavor to select such as are believed to be new, or as seem to possess special interest, and will try to grade them so as to suit the different degrees of advancement of our readers. The solutions to mathematical questions will, in general, be published in the second No. succeding the one in which the questions are published.

ON THE RELATIVE POSITION OF THE ASTEROIDAL ORBITS.

BY PROF. DANIEL KIRKWOOD.

The *Annuaire du Bureau des Longitudes pour l'an* 1873 contains the elements of the orbits of 115 minor planets. The mean number of perihelia for every 15° of longitude is therefore 4.79. It is proposed to inquire whether any marked irregularity obtains in their distribution around the ecliptic.

Of the 115 asteroids in the table only 27 have their perihelia between 150° and 300° of longitude. This is a mean of 2.7 for evry 15° of arc; while the average number for every 15° of the remaining 210° is 6.29.

Again: a similar irregularity is found in the postion of the ascending nodes; the region of sparse distribution being less extensive than the former. but included within it. Thus; between 225° and 285° we find but 5 ascending nodes, or 1.25 for 15°; while the mean for the remaining 300° is 5.5 for

each arc of 15°. Is this striking disparity merely accidental? or has it resulted from the operation of a physical cause?

The fact may perhaps be sufficiently explained by the remark of Prof. Newcomb that "there is always a tendency in the perihelia of the asteroids to coincide in longitude with the perihelion of Jupiter and in their nodes to coincide in longitude with the nodes of Jupiter."

THE RECURRENCE OF ECIPSES.

BY PROF. DAVID TROWBRIDGE, WATERBURGH, N. Y.

That eclipses recur in the same order in a cycle of about eighteen years was known to the ancient Chaldeans, who probably discovered the period from observations, by comparing together the records of maney eclipses. This period, which they called the *saros*, must have been of great advantage to the ancient astronomer in predicting eclipses; since a record of all the solar eclipses (on an average about 41), and of all the lunar eclipses (on an average about 29) in the order in which they occurred, during any one of the complete cycles, would enable him to predict approximately the eclepses of the next succeeding period or cycle. The coincidences required, however, are not sufficiently exact to give more than approximate results; and if there are several intervening cycles, the recurrence is not very reliable even as an approximation. I have never seen in any astronomical work any other periods referred to,(though it is quite possible that *some* work may contain such reference), though other and much more exact periods exist, and one of them only about three times the length of the *saros*, as I shall now show.

According to Bessel the length of the sidereal year is 365.2563582 mean solar days.

The mean sidereal revolution of the moon's nodes is equal to 6793.39108 mean solar days.

The revolution of the moon's nodes being accomplished in a direction opposite to the apparent revolution of the sun, if they set out together at any time, they will again come togeiher in less than a year; or really in 346.619849 mean solar days. This is called the mean (as all these periods are mean periods) synodical revolution of the moon's nodes. The mean synodical revolution of the moon, or the period from one new moon to the next succeeding new moon, is 29.5305887 mean solar days. The several approximate ratios of these last two numbers will make known to us the

time required for the sun, moon and nodes, all setting out from the same point in the heavens, to return, approximately, to the same relative mean position. If we reduce the ratio of these periods to a continued fraction, we shall have

$$\frac{346.6198480}{29.5305887}=11+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{4+\cfrac{1}{3+\cfrac{1}{5+\cfrac{1}{47+.}}}}}}}$$

The several approximate ratios, or the sums of the partial fractions, are

$\frac{11}{1}$, $\frac{12}{1}$, $\frac{35}{3}$, $\frac{47}{4}$, $\frac{223}{19}$, $\frac{716}{61}$, $\frac{3803}{324}$, $\frac{179457}{15289}$, &c.

The fifth ratio in this series is that known as the *saros*. The sixth and seventh, so far as I know, have not been given before, and the eighth one is too long to be useful. The errors of the periods are as follows:

$29^d.5305887\times223=6585^d.32128=18^y$ (of 365 days) and $15^d.231$.
$346^d\cdot619848\ \times19\ =6585^d\cdot77711$.
Difference, $o^d.45583=10^h.94$.

$29^d.5305887\times716=21143^d.901509=57^y$. (of 365 days) and $338^d.901$.
$346^d.619848\ \times61\ =21143^d.810728$.
Difference, $0^d.090781=2^h.1787$.

$29^d.5305887\times3803=112304^d.828826=307^y$(Of 365 days) and 249.828
$346^d.619848\ \times324\ =112304^d.830752$.
Difference, $0^d.001926=0^h2^m.773$.

Although these numbers are the mean values, yet the true values will only change the character of the eclipse, and not prevent it from taking place. This is especially true in the periods of 57 and 307 years. Ten successive recurrences of the 307-year cycle will not be half an hour from exact coincidence. It seems, therefore, that this period will serve as a check in computing the time when ancient eclipses happened.

A total eclipse of the sun is referred to by Herodotus,which has been the subject of much discussion. Baily placed it by his calculations on the 30th of September, 610 B. C. Eight times the 307-year cycle brings us to the 28th of July, 1851, the great total eclipse of that year.

Prof. Airy's calculations fixed the time of the eclipse on the 28th of may,

585 B. C. Eight times the 307-year cycle brings us to April 24, 1876. It does not appear that any eclipse will happn at that time.

Another eclipse that seems to be referred to by Xenophon happened according to Prof. Airy, May 19, 556 B. C. Eight 307-year periods, minus one 57-year period, brings us to April 27, 1847. No eclipse happened at that time.

The eclipse of March 20th, 1140 A. D., total at London, returned December 21, 1870, after two 307-year periods, plus two 57-year periods. The eclipse of August 21, 1560, returned February 22, 1868, after one 307 year period; and the eclipse of April 9, 1567, returns October 9, 1874, after one 307-year period.

Of instances of the 57-year period we may mention that the eclipse of July 14, 1748, returned in 1806, June 16th; and this last returned May 5, 1864. The great eclipse of June 24, 1778, returned in 1836, May 15th, as an annular eclipse, after one period.

So far as I have been able to compare these periods with eclipses whose dates are certain, they give the order of the eclipses.

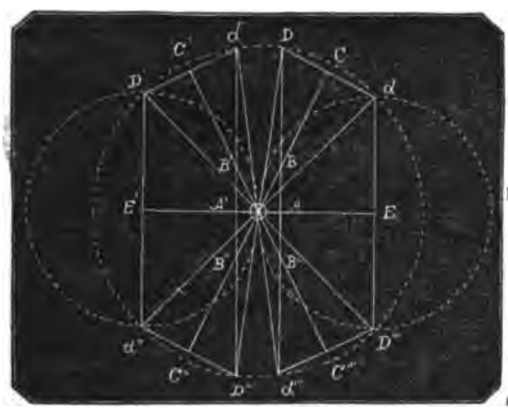
OPERATIONS ON IMAGINARY QUANTITIES CONSIDERED GEOMETRICALLY.

BY PROF. W. D. HENKLE, SALEM, OHIO.

In the view here taken of so-called imaginary quantities the $\sqrt{-1}$ is not considered as a quantity at all, but merely as a sign of an operation. First, let us consider the multiplication of binomial expressions. The following table of signs shows that there are sixteen cases:—

+	+	+	+	+	+	+	+
+	+	+	—	—	+	—	—
+	—	+	—	+	—	+	—
+	+	+	—	—	+	—	—
—	+	—	+	—	+	—	+
+	+	+	—	—	+	—	—
—	—	—	—	—	—	—	—
+	+	+	—	—	+	—	—

[1]	[2]	[3]	[4]
$1+2\sqrt{-1}$	$1+2\sqrt{-1}$	$1+2\sqrt{-1}$	$1+2\sqrt{-1}$
$3+\sqrt{-1}$	$3-\sqrt{-1}$	$-3+\sqrt{-1}$	$-3-\sqrt{-1}$
$1+7\sqrt{-1}$	$5+5\sqrt{-1}$	$-5-5\sqrt{-1}$	$-1-7\sqrt{-1}$
[5]	**[6]**	**[7]**	**[8]**
$1-2\sqrt{-1}$	$1-2\sqrt{-1}$	$1-2\sqrt{-1}$	$1-2\sqrt{-1}$
$3+\sqrt{-1}$	$3-\sqrt{-1}$	$-3+\sqrt{-1}$	$-3-\sqrt{-1}$
$5-5\sqrt{-1}$	$1-7\sqrt{-1}$	$-1+7\sqrt{-1}$	$-5+5\sqrt{-1}$
(9)	**(10)**	**(11)**	**(12)**
$-1+2\sqrt{-1}$	$-1+2\sqrt{-1}$	$-1+2\sqrt{-1}$	$-1+2\sqrt{-1}$
$3+\sqrt{-1}$	$3-\sqrt{-1}$	$-3+\sqrt{-1}$	$-3-\sqrt{-1}$
$-5+5\sqrt{-1}$	$-1+7\sqrt{-1}$	$+1-7\sqrt{-1}$	$+5-5\sqrt{-1}$
(13)	**(14)**	**(15)**	**(16)**
$-1-2\sqrt{-1}$	$-1-2\sqrt{-1}$	$-1-2\sqrt{-1}$	$-1-2\sqrt{-1}$
$2+\sqrt{-1}$	$3-\sqrt{-1}$	$-3+\sqrt{-1}$	$-3-\sqrt{-1}$
$-1-7\sqrt{-1}$	$-5-5\sqrt{-1}$	$5+5\sqrt{-1}$	$+1+7\sqrt{-1}$



In the first example, take O A =1 and A B=2 at right angles to O A, in consequence of the sign of operation $\sqrt{-1}$ with which 2 is affected; then O B represents $12+\sqrt{-1}$ in magnitude and position.

Ttake O C=3 O B; then O C represents 3 times $1+2\sqrt{-1}$. Draw CD=OB at right angles to OC; then CD represents $\sqrt{-1}$ times $1+2\sqrt{-1}$ as added to 3 times $1+2\sqrt{-1}$. Hence OD represents the full product of $1+2\sqrt{-1}$ multiplied by $3+\sqrt{-1}$. It is easily proved that AB produced passes through D and that AD=7. Hence OD is also the construction of $1+7\sqrt{-1}$.

In the second example the $\sqrt{-1}$ of the multiplier being negative, C*d*= OB is drawn in the opposite direction from CD. this gives O*d* as the final product. Draw *d*E perpendicular to OE. It is easily proved that OE and *d*E are each equal to 5. Hence O*d* is the construction of $5+5\sqrt{-1}$

In the third example, the multiplier may be placed under the form $-(+3-\sqrt{-1})$; hence the result is the same as the result in the second example taken negatively. O*d''* equals O*d* taken negatively. It is easily proved that OE′=and E′*d''* are each equal to 5. Hence O*d''* is the construction of—$5-5\sqrt{/}1$.

It is plain that in the fourth eaxmple the multiplier differs from that in the first only in being negative. Hence the result is O D'' or O D taken negatively. It is also evident that OD'' is the construction of $-1-7\sqrt{-1}$.

In the fifth example, take OA=1 and $AB'''=2$, then OB''' is the construction of $1-2\sqrt{-1}$. Take $OC'''=3$ OB''' anc draw $C'''D'''$ at right angles to OC'''. The final product is OD'''. This is also the construction of $5-5\sqrt{-1}$, since OE and ED''' can be proved to be each equal to 5.

In the sixth example $C'''d'''=OB'''$ is drawn on the other side of OC''' since $\sqrt{-1}$ is negative. The final product is Od''', whiah is also the construction of $1-7\sqrt{-1}$.

In the seventh example place $-3+\sqrt{-1}=-(+3-\sqrt{-1})$. The result must then be the same as that in the sixth taken negatively, that is Od' which is $-Od'''$. Od' is also the cohstruction of $-1+7\sqrt{-1}$.

In the eighth example place $-3-\sqrt{-1}=-(+3+\sqrt{-1})$. Hence the result must be the negative of that in the fifth, that is $-OD'''$, or OD'. OD' is also the construction of $-5+5\sqrt{-1}$.

In the ninth example, take $OA'=-1$ and $A'B'=2$; then is OB' the construction of $-1+2\sqrt{-1}$. Take $OC'=3OB'$ and draw $C'D'=OB'$. The final product OD' is also the construction of $-5+5\sqrt{-1}$.

In the tenth example, draw $C'd'=OB'$. The final result Od' is also the construction of $-1+7\sqrt{-1}$.

In the eleventh example, place $-3+\sqrt{-1}=-(+3-\sqrt{-1})$. Hence the final product must be the negative of that in the tenth, or Od''', which is also the construction of $1-7\sqrt{}1$.

In the twelfth example, place $-3-\sqrt{-1}=-(+3+\sqrt{-1})$. Hence the final product is the negative of that in the ninth, or OD''', which is also the construction of $5-5\sqrt{-1}$.

In the thirteenth example take $OA'=-1$ and $A'B''=2$, then is OB'' the construction of $-1-2\sqrt{-1}$. Draw $OC''=3OB''$ and $C''D''=OB''$. Hence the final product is OD'', which is also the construction of $-1-7\sqrt{-1}$.

In the fourteenth example, draw $C''D''=OB''$. Hence the final product is Od'', which is also the construction of $-5-5\sqrt{-1}$.

In the fifteenth example place $-3+\sqrt{-1}=-(+3-\sqrt{-1})$. Hence the final product is the negative of that in the fourteenth, or Od, which is also the construction of $5+5\sqrt{-1}$.

In the sixteenth, place $-3-\sqrt{-1}=-(+3+\sqrt{-1})$. Hence the product is the negative of that in the thirteenth, or OD, which is also the construction of $1+7\sqrt{-1}$.

The conventions adopted in these illustrations are the simplest that have occurred to me.

I may in a future articl illustrate division from the same diagram, and addition and subtraction from other diagrams.

In the mean time let those who can, give a geometrical illustration of an imaginary exponent.

EQUATIONS OF DFFEREINCS.

BY WALTER SIVERLY, OIL CITY, PA.

I am not aware that this subject has been discussed in any American work, and, so far as I know, in but few foreign ones. The only works that I have seen that treat upon this subject are "Hymer's Finite Differences" and "De Morgan's Differential and Integral Calculus."

The limits of this article will not allow me to introduce the discussions involved in the integration of the equations; but I will give, 1st, the formula resulting from the integration of an equation of the first degree, in the first order of differences, with an example of its application; and, 2nd, the formula resulting from the integration of an equation of the n^{th} degree, with an example also of its applicatio. These formulæ can be used without further knowledge of the subject.

1. Let u_0, u_1, u_2, . . . u_{x-1}, u_x, u_{x+1}, &c., be any series of whic u_x, a function of x, is the general term; any relation between the terms in functions of x expressed by means of an equation, is called an equation of differences.

Let the relation be $u_{x+1}—Au_x=B$, A and B being constants. The integration of this equation gives $u_x=CA^x+\dfrac{B}{1+A}$, ... (1)

C being an arbitrary constant determined as in ordinary integration.

Example.—One of two casks contains a gallons of wine and the other b gallons of water; c gallons are taken from the first and poured into the second cask, and then c gallons are taken from the second and poured into the first. Required the quantity of wine in the second cask after n such operations.

Let $u_n=$ the wine in the second cask after the n^{th} operation, the first cask will then contain $a—u_n$ gallons of wine; at the next operation there will be taken out of the first cask

$$\frac{c}{a}(a—u_n)$$

gallons of wine, which being poured in to the second cask it will then con tain

$$u_n + \frac{c}{a}(a-u_n) = c + \left\{1-\frac{c}{a}\right\} u_n$$

gallons of wine; but the second cask now contains $b + c$ gallons in all. Hence, taking out c gallons of the mixture in the second cask, there will remain in the second cask

$$\frac{b}{b+c}\left\{c+\left(1-\frac{c}{a}\right)u\right\} \text{ gallons of wine.}$$

$$\therefore \frac{b}{b+c}\left\{c+\left(1-\frac{c}{a}\right)u_n\right\} = u_{n+1}.$$

$$\therefore u_{n+1} - \frac{b}{a}\left(\frac{a-c}{b+c}\right)u_n = \frac{bc}{b+c}.$$

$$\therefore A = \frac{b}{a}\left(\frac{a-c}{b+c}\right) \text{ and } B = \frac{bc}{b+c}. \therefore \text{ by (1), } u_n = C\left(\frac{b}{a}\right)^n\left(\frac{a-c}{b+c}\right)^n + \frac{ab}{a+b}.$$

$$\text{When } n=0,\ u_n=0,\ \therefore C = -\frac{ab}{a+b}.$$

$$\therefore u_n = \frac{ab}{a+b}\left\{1-\left(\frac{b}{a}\right)^n\left(\frac{a-c}{b+c}\right)^n\right\}.$$

2. In the equation $u_{x+n} + Au_{x+n-1} + Bu_{x+n-2} + \ldots + Pu_x = 0$, putting $u_x = z^x$ and substituting, the equation becomes $z^{x+n} + Az^{x+n-1} + Bz^{x+n-2} + \ldots + Pz^x = 0$. Dividing by z^x we have $z^n + Az^{n-1} + Bz^{n-2} + \ldots + P = 0$, an equation of n dimensions. Let the roots of this equation be $r_1, r_2, r_3, \ldots r_n$, then the complete integral of the original equation will be

$$u_x = C_1(r_1)^x + C_2(r_2)^x + C_3(r_3)^x + \ldots + C_n(r_n)^x \ldots\ldots (2)$$

C_1, C_2, C_3, &c., being n arbitrary constants.

EXAMPLE.—A heifer calf at the age of two years has a heifer calf, and one every year afterward. All the progeny increase in the same manner. How many will there be at the end of the x^{th} year?

Let u_x = the number at the end of the x^{th} year. The increase for any year is from all those of two years preceding and from no others, hence

$$u_{x+2} - u_{x+1} = u_x.$$

But the first three terms of the series, u_0, u_1, u_2, &c,. are $u_0 = 1$, $u_1 = 1$, and $u_2 = 2$. Hence, putting $x = 0$, we have $A\,u_1 = 1$ and $Pu_0 = 1$, $\therefore A = -1$ and $P = -1$.

$$\therefore z^2 - z = 1; \therefore r_1 = \frac{1+\sqrt{5}}{2} \text{ and } r_2 = \frac{1-\sqrt{5}}{2}.$$

$$\therefore u_x = C_1\left(\frac{1+\sqrt{5}}{2}\right)^x + C_2\left(\frac{1-\sqrt{5}}{2}\right)^x.$$

When $x = 0$ $u_0 = C_1 + C_2 = 1$. And when

$$x = 1 \quad u_1 = C_1\left(\frac{1+\sqrt{5}}{2}\right) + C_2\left(\frac{1-\sqrt{5}}{2}\right) = 1.$$

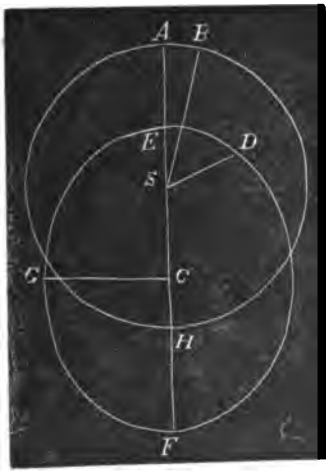
$$\therefore C_1 = \frac{5+\sqrt{5}}{10}, \text{ and } C_2 = \frac{5-\sqrt{5}}{10}.$$

$$\therefore u_x = \frac{(5+\sqrt{5})(1+\sqrt{5})^x + (5-\sqrt{5})(1-\sqrt{5})^x}{10(2)^x}.$$

PROBLEM.—TO FIND THE RELATION BETWEEN THE MEAN ANOMALY AND THE TRUE ANOMALY.

BY PROF. JOSEPH FICKLIN, COLUMBIA, MO.

Let E D F be the orbit of the planet, having the sun in the focus at S.



Put a = C E, b = C G, and from S as a center and a radius S A equal to $\sqrt{(ab)}$, describe the circumference of a circle; then the circle and the ellipse will contain equal areas.

At the same time that the planet departs from E, the perihelion, let a body begin to move with a uniform motion from A through the circumference A B H, and perform a whole revolution in the same time that the planet describes the ellipse. Suppose the body describing the circle to be at B when the planet is at D, the angle A S B is the mean anomaly, and E S D the true anomaly.

Put x = the angle E S D, and θ = A S B; then the area of A S B is

$$\frac{AS^2}{2}\theta = \frac{ab}{2}.\theta.$$

Put r = radius-vector of the ellipse, then the area of E S D will be

$$\frac{1}{2}\int r^2 dx; \therefore \frac{ab\theta}{2} = \frac{1}{2}\int r^2 dx, \text{ or } ab\theta = \int r^2 dx. \ldots (1)$$

The polar equation of the ellipse, the focus being at the pole, is

$$r = \frac{p}{1+e\cos x}; \therefore \int r^2 dx = p^2 \int \frac{dx}{(1+e\cos x)^2} \ldots\ldots (2)$$

Assume $\int \frac{dx}{(1+e\cos x)^2} = \frac{A\sin x}{1+e\cos x} + B\int \frac{dx}{1+e\cos x}$.

Taking the differential coefficients, we have

$$\frac{1}{(1+e\cos x)^2} = \frac{A\cos x\,(1+e\cos x)+Ae\sin^2 x}{(1+e\cos x)^2} + \frac{B(1+e\cos x)}{(1+e\cos x)^2};$$

$$\therefore 1 = A\cos x\,(1+e\cos x) + Ae\sin^2 x + B\,(1+e\cos x)$$

$$= A\cos x + Ae\cos^2 x + Ae\sin^2 x + B + Be\cos x = (A+Be)\cos x + Ae + B.$$

Equating like powers of $(\cos x)$, we have $A+Be=0$, and $Ae+B=1$ whence

$$B = \frac{1}{1-e^2}, \text{ and } A = -\frac{e}{1-e^2};$$

$$\therefore \int \frac{dx}{(1+e\cos x)^2} = -\frac{e}{1-e^2}\cdot\frac{\sin x}{1+e\cos x} + \frac{1}{1-e^2}\int\frac{dx}{1+e\cos x}.$$

Put $y = \cos x$, then

$$dx = -\frac{dy}{\sqrt{(1-y^2)}}, \text{ and} \int \frac{dx}{1+e\cos x} = -\int \frac{dy}{(1+ey)\sqrt{(1-y^2)}}.$$

Again, put $1-y^2 = (1-y)^2 z^2$;

$$\therefore y = \frac{z^2-1}{z^2+1}, dy = \frac{4zdz}{(z^2+1)^2}; \therefore -\int \frac{dy}{(1+ey)\sqrt{(1-y^2)}}$$

$$= -2\int \frac{dz}{1-e+z^2(1+e)} = -\frac{2}{1-e}\int \frac{dz}{1+\left(\frac{1+e}{1-e}\right)z^2}$$

$$= -\frac{2}{(1-e)\sqrt{\left(\frac{1+e}{1-e}\right)}} \int \frac{\sqrt{\left(\frac{1+e}{1-e}\right)}\,dz}{1+\left\{\sqrt{\left(\frac{1+e}{1-e}\right)}\,z\right\}^2}$$

$$=-\frac{2}{\sqrt{(1+e)(1-e)}}\int\frac{\sqrt{\frac{1+e}{1-e}}.\,dz}{1+\left(\sqrt{\frac{1+e}{1-e}}.\,z\right)^2}$$

$$=-\frac{2}{\sqrt{1-e^2}}\int\frac{\sqrt{\frac{1+e}{1-e}}.\,dz}{1+\left(\sqrt{\frac{1+e}{1-e}}.\,z\right)^2}=-\frac{2}{\sqrt{1-e^2}}.\tan^{-1}\left(\sqrt{\frac{1+e}{1-e}}.\,z\right).$$

$\therefore$ area $E\,S\,E$

$$=\frac{1}{2}\int r^2 dx=-\frac{p^2\,e\sin x}{2(1-e^2)(1+e\cos x)}-\frac{2p^2\,.\tan^{-1}\left(\sqrt{\frac{1+e}{1-e}}.\,z\right)}{2(1-e^2)\sqrt{1-e^2}}+C$$

$$=-\frac{p^2\,e\sin x}{2(1-e^2)(1+e\cos x)}-\frac{p^2}{(1-e^2)^{\frac{3}{2}}}.\tan^{-1}\left(\sqrt{\frac{1+e}{1-e}}.\,z\right)+C$$

$$=-\frac{p^2\,e\sin x}{2(1-e^2)(1+e\cos x)}-\frac{p^2}{(1-e^2)^{\frac{3}{2}}}.\tan^{-1}\sqrt{\frac{1+e}{1-e}.\frac{1+y}{1-y}}+C$$

$$=-\frac{p^2\,e\sin x}{2(1-e^2)(1+e\cos x)}-\frac{p^2}{(1-e^2)^{\frac{3}{2}}}.\tan^{-1}\sqrt{\frac{1+e}{1-e}.\frac{1+\cos x}{1-\cos x}}+C$$

$$=-\frac{p^2\,e\sin x}{2(1-e^2)(1+e\cos x)}-\frac{p^2}{(1-e^2)^{\frac{3}{2}}}.\tan^{-1}\sqrt{\frac{1+e}{1-e}.\frac{2\cos^2\frac{1}{2}x}{2\sin^2\frac{1}{2}x}}+C$$

$$=-\frac{p^2\,e\sin x}{2(1-e^2)(1+e\cos x)}-\frac{p^2}{(1-e^2)^{\frac{3}{2}}}.\tan^{-1}\left(\sqrt{\frac{1+e}{1-e}}.\cot\tfrac{1}{2}x\right)+C.$$

When $x=0$, area $E\,S\,D=0$; $\therefore$ from this equation

$$0=-\frac{p^2}{(1-e^2)^{\frac{3}{2}}}.\frac{\pi}{2}+C;\ \text{ whence } C=\frac{p^2\,\pi}{2(1-e^2)^{\frac{3}{2}}},$$

and therefore the general expression for the area $E\,S\,D$ is

$$\frac{1}{2}\int r^2\,dx=-\frac{p^2\,e\sin x}{2(1-e^2)(1+e\cos x)}-\frac{p^2}{(1-e^2)^{\frac{3}{2}}}.\tan^{-1}\left(\sqrt{\frac{1+e}{1-e}}\cot\tfrac{1}{2}x\right)+\frac{p^2\,\pi}{2(1-e^2)^{\frac{3}{2}}}.$$

Substituting this value of $\int r^2\,dx$ in (1) we obtain

$$a\,b\,\theta=\frac{p^2\,\pi}{(1-e^2)^{\frac{3}{2}}}-\frac{p^2\,e\sin x}{(1-e^2)(1+e\cos x)}-\frac{2p^2}{(1-e^2)^{\frac{3}{2}}}.\tan^{-1}\left(\sqrt{\frac{1+e}{1-e}}.\cot\tfrac{1}{2}x\right);$$

but $p^2 = \frac{b^4}{a^2}$, and $1—e^2 = \frac{b^2}{a^2}$, or, $(1—e^2)^{\frac{3}{2}} = \frac{b^3}{a^3}$; $\therefore \frac{p^2}{(1—e^2)^{\frac{3}{2}}} = \frac{a^4}{b^2} \div \frac{b^3}{a^3}$

$= a\,b$, and the above equation becomes

$$a\,b\,\theta = a\,b\,\pi — \frac{b^2\,e \sin x}{1 + e \cos x} — a\,b \tan^{-1}\left(\sqrt{\frac{1+e}{1—e}} \,.\, \cot \tfrac{1}{2}\,x\right),$$

$$\text{or, } a\,\theta = a\,\pi — \frac{b\,e \sin x}{1+e \cos x} — a \tan^{-1}\left(\sqrt{\frac{1+e}{1—e}} \,.\, \cot \tfrac{1}{2}x\right),$$

$$\text{or, } \theta = \pi — \frac{be \sin x}{a(1+e \cos x)} — \tan^{-1}\left(\sqrt{\frac{1+e}{1—e}} \,.\cot \tfrac{1}{2}\,x\right) \;\ldots\ldots\; (3);$$

and this is the relation existing between the mean anomaly and the true anomaly.

SOLUTION OF A QUESTION INVOLVING A MINIMUM.

The following problem is found in the *Mathematical Monthly*, (Runkle's): "Find a point O, within a triangle, such that $O\,A^n + O\,B^n + O\,C^n =$ a minimum."

[The solution there given is a good one; but a friend of mine, in the United States Coast Survey, dreamed out the solution below, and conveyed it to paper in the morning. I give it for publication; but he disallows the use of his name.]

[*Notation seen from the figure.*]

$x^n + y^n + z^n =$ a minimum.

Let x at first be constant, y anc

z vary.

$\therefore y^{n-1} d\,y + z^{n-1} d\,z = 0 \ldots\ldots (1$

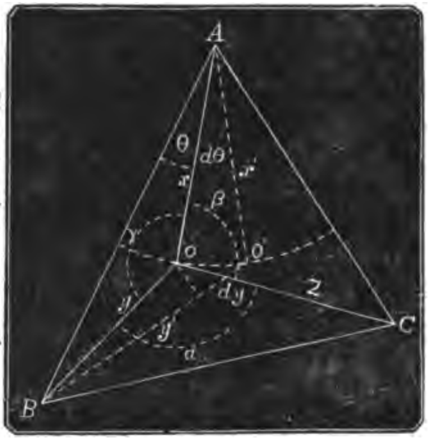
Now $dy = xd\theta \sin \gamma$, from infinitesi-mal triangle,

and similarly $dz = — xd\theta \sin \beta$.

Substituting in (1)

$xd\theta(y^{n-1} \sin \gamma — z^{n-1} \sin \beta) = 0.$

$\therefore \frac{y^{n-1}}{\text{siu } \beta} = \frac{z^{n-1}}{\sin \gamma}.$



If x also vary we have $\frac{x^{n-1}}{\sin \alpha} = \frac{y^{n-1}}{\sin \beta} = \frac{z^{n-1}}{\sin \gamma}.$

If $n = 1$, $\alpha = \beta = \gamma$, each $= 120°$. If $n = 2$, $\frac{x}{\sin\alpha} = \frac{y}{\sin\beta} = \frac{z}{\sin\gamma}$, that is, the point is the center of gravity of the triangle.

THEO. L. DE LAND, TREAS. DEP'T, WASHINGTON, D. C.

REPETENDS.

BY PROF. M. C. STEVENS, SALEM, OHIO.

In the February number of the *Michigan Teacher* for 1869, there appeared an article, contributed by Mr. William Wiley of Detroit, entitled, "New Theory of Repetends," in which it is shown that the figures of a repetend are easily deduced from the common fracton successively from right to left, instead of from left to right as is done by the ordinary method.

Thinking it deserving of more notice than it received from its publication in that journal, I propose in this article to reproduce, in substance, Mr. Wiley's method, and also demonstrate some of the curious properties of repetends.

First Principle.—In every common fraction which reduces to a pure repetend the unit's figure of the denominator must be either 1, 3, 7, or 9.

Second Principle.—If the numerator of the fraction be unity, the last figure of the repetend is either 9, 3, 7, or 1, respectively, according as the units of the denominator is 1, 3, 7, or 9.

Third Principle.—If $\frac{1}{d}$ be the fraction that reduces to a repetend, q_1 the last figure before it commences to repeat, r_1 the last and r_2 the next to the last remainder, then evidently

$$r_1 = 1, \text{ and } q_1 = \frac{10r_2 - 1}{d}; \text{ whence } r_2 = \frac{dq_1 + 1}{10} \quad . \quad . \quad . \quad . \quad (1)$$

Fourth Principle.—If $q_n, q_{n-1} \ldots\ldots q_2$ and q_1 be the digits of the repetend, then is

$$\frac{1}{d} = \frac{10^{n-1}q_n + 10^{n-2}q_{n-1} + \ldots 10q_2 + q_1 + \frac{1}{d}}{10^n} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\text{and } \frac{r_2}{d} = \frac{10^{n-1}q_1 + 10^{n-2}q_n + \ldots 10q_3 + q_2 + r_2 \div d}{10^n} \quad . \quad . \quad . \quad . \quad (3)$$

in which we have the same sequence of digits, the last taking the first place and the remaining figures being each removed one place to the right.

For an example, take $\frac{1}{7} = .142857\frac{1}{7}$; then $\frac{5}{7} = .714285\frac{5}{7}$. This is evident,

since $\frac{1}{7}=.14285\frac{5}{7}$, and the value of the fraction at the end is five times the, part found, $\frac{5}{7}=.71428\frac{4}{7}$; whence $\frac{1}{7}=.14285714428\frac{4}{7}=.142857142856\frac{5}{7}$. We thus see the necessity of the sequence above stated.

When we multiply (2) by r_2, if we represent the tens of r_2q_1, r_2q_2, &c., by m_1, m_2, &c., we evidenely have

$$r_2q_1 = 10m_1+q_2 \quad \text{(1')}$$
$$r_2q_2+m_1 = 10m_2+q_3 \quad \text{(2')}$$
$$r_2q_3+m_2 = 10m_3+q_4 \quad \text{(3')}$$
&c., &c.

We here have the key to Mr. Wiley's method; for by the *second principle* q_1 is known at a glance, and by the *third principle* r_2 is found by equation (1); whence q_2 becomes known from (1′), and q_3, q_4, &c., are successively found from equations (2′), (3′), &c.

(*To be continued.*)

PROBLEMS.

1. Find the value of x and y in the following equations:

$$a^2x^4 + b^2y^4 = a^2b^2(x+y)^2;$$
$$a^2x^2 + b^2y^2 = a^2b^2.$$

—Communicated by U. Jesse Knisely, Pres't and Prof. of Mathematics in *Luther College*, Newcomerstown, Ohio.

2 Let a regular polygon of 14 sides be described, each of whose equal sides shall be *one*. Then will the radius of its circumscribing circle, which put$=r$, be more than *two* and less than *three*. Put $r = 2 + x$; then is x a positive quantity less than *one*. Let another regular polygon of half the number of sides (7) be inscribed in a circle whose radius is *one*, and determine one of its equal sides in functions of x expressed in its simplest form.

3. If a line make an angle of 40° with a fixed plane, and a plane embracing this line be perpendecular to the fixed plane, how many degrees from its first position must the plane embracing the line revolve in order that it may make an angle of 45° with the fixed plane?—Communicated by Prof. A. Schuyler, Berea, Ohio.

4. A cask contrining a gallons of wine stands on another containing a gallons of water; they are connected by a pipe, through which, when open, the wine can escape in to the lower cask at the rate of c gallons per minute, and through a pipe in the lower cask the mixture can escape at the same rate; also, water can be let in through a pipe on the top of the upper cask

at a like rate. If all the pipes be opened at the same instant, how much *wine* will be in the lower cask at the end of t minutes, supposing the fluids to mingle perfectly?—Communicated by ARTEMAS MARTIN, Mathematical Editor of *Schoolday Magazine*, Erie, Pa.

NOTE.— To those who use "Nystrom's Mechanics." Nystrom prints

"$\pi^2 = 9.869650000+$,"

but $\pi^2 = 9.86960440108+$,—U. JESSE KNISELY.

QUERY.—What is the explanation of the phenomena described below?

If a ball of cork or other light substance be placed in a vertical jet of water of sufficient force to elevate the ball, it will rise to a point where the force of the ascending jet, or so much of it as is efficient in elevating the ball, is just equal to the weight of the ball, and will there *revolve*; and its equilibrium will continually be restored, notwithstanding the ball may be disturbed by slight horizontal forces.

Erratum. On page 6, 19th line from bottom, for "$12 + \sqrt{-1}$" read $1 + 2\sqrt{-1}$.

BOOK NOTICES.

Comets and Meteors. By DANIEL KIRKWOOD, L. L. D., Professor of Mathematics in Indiana University, and author of "*Meteoric Astronomy*." J. B. Lippincott & Co., Philadelphia.

To those who have not yet seen this very interesting book by Prof. Kirkwood, the follow ing quotation from the *Preface* will serve to indicate its character:

"The origin of meteoric astronomy, as a sciene, dates from the memorable star shower of 1833. Soon after that briliant display it was found that similar phenomena had been witnessed, at nearly equal intervals, in former times. This discovery led at once to another no less importrnt, viz: that the nebulous masses from which such showers are derived revolve around the sun in paths intersecting the earth's orbit. The theory that these meteor-clouds are but the scatered fragments of disintegrated comets was announced by several astronomers in 1867—a theory confirmed in a remarkable manner by the shower of meteors from the *debris* of Biela's comet on the 27th of November 1872. To gratify the interest awakened in the public mind by the discoveries here named, is the object of this work. Among the subjects considered are, cometary astronomy; aerolites, with the phenomena atteuding their fall; the most briliant star-showers of all ages, and the origin of comets aerolites and falling stars.'

Surveying and Navigation, with a Preliminary Treatise on Trigonometry and Mensuration.' By A. SCHUYLER, A. M., Professor of Applied Mathematics and Logic in Baldwin University; author of "*Higher Arithmetic*," "*Principles of Logic*," and "*Complete Algebra.*' Wilson, Hinkle & Co., Cincinnati and New York.

We would be pleased to give an extended notice of this book did our space permit. We must be content to say, however, that, as a text book for the student, and as a manual for the surveyor, we think it admirable, both in plan and execution. The subjects discussed are thoroughly and yet concisely dealt with; and the paper, wood cuts and typography are perfect.

Yates County Chronicle. Persons who are fond of solving mathematical problems and who want something new on that subject every week and a good newspaper besides, will do well to obtain the *Yates County Chronicle*; published at Penn Yan, New York. DR. S. H. WRIGHT, Mathematical Editor.

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EDITED AND PUBLISHED BY

J. E. HENDRICKS, A. M.

CONTENTS:

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NOTICE.—The first number of THE ANALYST was sent to persons only who were known as mathematicians, or who had signified their willingness to examine a specimen number and return it if not desired. A very considerable number ef copies have been returned, most of them marked, so that the persons who returned them could be identified; a few, however, have been returned which contain no marks from which the names of the persons who returned them could be determined.

As there has been no response yet from a majority of those to whom the first number was sent, nor a return of the number, the second number will be sent to some who have not yet signified their intention to become subscribers, and in all such cases it will be sent in a *paid* wrapper. Persons, therefore, who may receive the second number in a paid wrapper, and who do not wish to become subscribers, will please return the number with their *names* upon it. And if any person who has returned the first number should receive the second number he will please understand that it is because the returned number was not identified as his, and not through any desire on the part of the publisher to importune his patronage.

All who receive the second number in an *unpaid* wrapper will understand that their subscriptions have been received and properly credited.

J. E. HENDRICKS.

THE ANALYST.

Vol. I. February, 1874. No. 2.

COMETS AND METEORS.

BY PROF. ASAPH HALL, WASHINGTON, D. C.

In bringing forward the nebular hypothesis to explain the cause of the primitive movements of our solar system Laplace states the following five phenomena which must be considered. 1. The motions of all the planets in the same direction and very nearly in the same plane: 2. The motions of the satellites in the same direction as that of the planets: 3. The motions of rotation of these different bodies and of the sun in the same direction as their projected motions, and in nearly the same planes: 4. The small excentricities of the orbits of the planets and their satellites: 5, and finally the great excentricities of the orbits of comets, while the inclinations of these orbits seem to occur wholly at random.

The only previous hypothesis to which Laplace refers is that of Buffon, the naturalist. Buffon assumed that a comet falling upon the sun had thrown out a torrent of matter, which, uniting at different distances into various globes, had become opaque and solid by cooling, and thus formed the planets and their satellites. This hypothesis would explain the first of the five preceding phenomena stated by Laplace, for all the bodies thus formed would move very nearly in the plane passing through the center of the sun and the direction of the torrent; but the four other phenomena cannot be explained by it. In fact, the smallness of the excentricities of the planetary orbits is directly opposed to this hypothesis; for if a body moving in an ellipse touches the sun it will do so at each of its revolutions, and, although this condition of things might be somewhat modified in Buffon's hypothesis, the chance that the excentricities of the orbits would be small is very slight. Finally, this hypothesis does not account for the comets at all. Among those who preceded Laplace in speculations on the constitution of the universe, after the discovery of the law of gravitation, may be mentioned Kant, the metaphysician, Lambert and Sir William Herschel. In a work published in 1755 under the title

of "A General Natural History and Theory of the Heavens," Kant undertakes the discussion of no less a problem than this: "To discover the arrangement which unites the great members of the creation in all its parts, even to infinity, and to deduce, by the aid of mechanical laws, the formation of the celestial bodies and the origin of their movements from the primitive state of nature." Kant's views are stated with clearness a d ingenuity, but Laplace, by means of his superior knowledge of astronomy, has made a much more definite statement of the phenomena to be explained, and his name is properly more closely connected with the nebular hypothesis than that of any one else. In the hands of Laplace this hypothesis gives a plausible explanation of the formation of the planets of our system and of their satellites; of the zodiacal light and of the singular equality observed between the angular motions of rotation and revolution of the satellites. He explains also the remarkable phenomenon presented by the first three satellites of Jupiter, and which consists in this, that the mean longitude of the first, minus three times that of the second, plus twice that of the third is always equal to two right angles. The chance is very small that such a condition should happen at random.

Under this hypothesis the comets are not considered as members of any planetary system, but are regarded as small nebulosities wandering with very small motions from one solar system to another, and formed by the condensation of nebulous matter lying near the limits of a system. Their motions being possible in all directions, when they pass under the control of the attracting force of our sun the inclinations of their orbits with respect to any plane should be distributed at random, and this accords very well with the observed fact. The great excentricities of the cometary orbits also result from this hypothesis; since, if their orbits are elliptical, the major axes must be at least equal to the radius of the sphere of activity of the attracting force of our sun. But from this hypothesis cometary orbits may be hyperbolic, and hence we may observe comets with a sensible hyperbolic movement. However, among the two hundred and thirty-eight comets whose orbits have been determined up to the present time, there is not a single one having a well determined hyperbolic orbit. Several such orbits have been computed, but in every case there is a good degree of probability that the observations can be satisfied within the limits of their probable errors by parabolic orbits. From this it appears that if comets enter our solar system they must do so in such a way that the chances are very slight for giving an orbit that is sensibly hyperbolic. Laplace submitted this question to the calculus of probabilities, and found

that there is six thousand to bet against one that a nebulosity that enters the sphere of activity of the sun, in such a way that it can be observed, will describe either a very elongated ellipse, or an hyperbola,which by the greatness of its axis may be taken for a parabola in the part where the comet is seen. In this solution it was assumed that the perihelion distance of the comet does not exceed twice the radius of the earth's orbit.

About the beginning of this century two German students began the systematic observation of shooting stars. For many years such observations promised but little, but within ten years past the astronomical theory of shooting stars and meteors has been greatly advanced, and many interesting relations have been discovered between the orbits of shooting stars and of comets. The comet is now generally regarded as the primitive body from whose gradual dissolution is formed a stream of small particles of matter moving around the sun in orbits similar to that of the comet. When the comet's orbit is situated very near the orbit of the earth some of these particles enter the atmosphere of the earth and being ignited by their passage through it produce the phenomenon of shooting stars. One of the most striking coincidences of the orbits of comets and of meteoric streams is that of the first comet of 1866 with the November stream of meteors. It was this coincidence that drew the attention of astronomers to the theory of these streams, although thirty years before Prof. Erman had pointed out the probable cosmical origin of meteoric streams and of their motions around the sun. Several other coincidences have been noticed, among the most interesting of which is the connection of a stream of meteors with the orbit of Biela's comet. Some observers make a distinction between shooting stars and meteors, the last, it is asserted, occurring more frequently during the early hours of the night when more of the meteoric stones have been seen to fall. But this distinction does not appear to be well establishad, since in the early part of the night there are many more observers, and again, the meteors coming into our atmosphere at that time of the night would have a smaller relative velocity and would be less likely to be consumed, and therefore would have a greater chance of reaching the earth's surface as solid bodies.

A few meteors have been observed which are remarkable on account of their very great velocities and their decided hyperbolic orbits. The following is a list of these so far as I have been able to collect cases well authenticated:

	Computer.	
1. Meteor of Oct. 29, 1857....	Petit....	Excentricity = 1.80, hyperbola doubtful.

		Computer.	
2.	Meteor of Nov. 15, 1860....	Newton..	No orbit computed, an hyperbola considered certain.
3.	Meteor of March 4, 1863....	Heis...	Excentricity=8.74
4.	Meteor of Jan. 30, 1868....	Galle...	Excentricity=2.28
5.	Meteor of Sept. 27, 1870....	Mattheissen...	Excentricity=1.182
6.	Meteor of June 17, 1873....	Niessl..	Excentricity=1.135, hyperbola doubtful.

To these results there is this objection; that they depend on observations that are very difficult to make, and which may be affected by such constant errors as will entirely vitiate the results; but most of the computers think there is but little doubt of the existence of hyperbolic orbits. Assuming this to be well established, it is further assumed that bodies moving about the sun with such velocities and in such orbits must come to us from the stellar regions. It is difficult to account for such velocities without making this assumption, and yet there are obvious objections to it. In the first place the chemical analysis of meteoric stones has given very nearly the same elements, and this points to a common origin. But assuming that these stones come to us from the stellar regions, it is certain that they came from the most diverse parts, and we are thus led to a vast assumption concerning the constitution of the matter of those regions. Again, if we assume that the meteors move in hyperbolic orbits and have nothing but an accidental connection with the earth's path in space, since they are visible only when they enter our atmosphere, there must be an immense number continually entering the sun's sphere of activity; for if we suppose that a comet becomes visible when it is at a distance from the sun equal to twice the earth's distance, and take the radius of the earth's sphere equal to 4,200 miles and compare the volume of the ring generated in space by the earth in one revolution to the volume of the sphere whose radius is twice the earth's distance from the sun the chance of visibility is nearly 800,000,000 times greater for a comet than for a meteor. This view, therefore, leads to the conclusion that there is continually going on an enormous interchange of matter between stellar systems, and as these bodies move in hyperbolic orbits it leads also to the result that such orbits must be the most probable, which is contradictory to the result obtained by Laplace.

The very great velocities which have been observed in the red flames forming the protuberances on the sun may lead to queries whether ejective and repelling forces may not act from the sun on certain kinds of matter connected with comets and meteors. This idea of a repulsive force acting

on certain particles of a comet and thus forming the tail was proposed by Olbers and first applied by Bessel in 1836 to his observations on the tail of Halley's comet. Bessel gave a complete discussion of the problem of determining the form of the tail under this assumption, together with a numerical application to his own observations. The same method has been applied by Prof. Peirce and Dr. Pape to the observations on the tail of Donati's comet in 1858, and by Prof. Schiaparelli to the observations of the comet of 1680. This method consists in applying to the particles that form the tail of the comet the common equations of motion and determining the constant that represents the force of the sun from the observations, assuming the law of force to be the inverse square of the distance. The following are the values of this constant found by the different computers, the plus sign denoting an attractive and the minus sign a repulsive force, the unit being the sun's attractive force at the distance unity:

Bessel......	Halley's comet, 1835...	force =	— 1.812
Peirce......	Donati's comet, 1858...	force =	— 1.500
Pape.......	Donati's comet, 1858...	force =	+ 0.612
Pape...	Donati's comet, 1858...	force =	— 5.317 secondary tail
Schiaparelli..	Donati's comet, 1680...	force =	0.000

The difference of more than two units in the results obtained by Prof. Peirce and Dr. Pape for the tail of the same comet may possibly be accounted for by the fact that in this case the force is determined from the observations with great uncertainty. However the results are not very satisfactory. If the numbers indicate anything they show that the sun acts on the particles of different comets with different forces, and also with different forces on the different tails of the same comet, or that the law of force has not been correctly assumed. While the phenomena of comet's tails indicate a repulsive force, the manner in which it acts seems to be unknown.

Prof. W. A. Norton in his investigations on the tail of Donati's comet has assumed that the repulsive force varies from one particle to another, and he finds the limits of this force to be —2.73 and +0.46. His conclusion is that the repulsion exerted by the sun, and also by the nucleus, is not a property belonging to all the particles of the mass, and he thinks that it is probably a magnetic or an electric force, emanating from the surface of the body or from a portion of its mass.

Leaving the preceding hypothesis of a repulsive force from the sun as having too little evidence to render it plausible under any known form, we shall find that a lunar or planetary origin of hyperbolic meteors has also

but a slight probability, and it appears necessary to return to the assumption that they originate in the stellar spaces. This is the position taken by Prof. Schiaparelli in his book on the "Astronomical Theory of Shooting Stars." But first he finds it necessary to remove the objection to the frequency of hyperbolic orbits, which results from the solution given by Laplace from the calculus of probabilities. Accordingly Prof. Schiaparelli gives a new solution of this problem and obtains a result directly opposite that of Laplace, finding that hyperbolic orbits are by far the most probable. It appears to me, however, that Prof. Schiaparelli is not successful in this attempt, and that in his solution he has omitted two considerations that must be attended to in a correct solution. If S be the sun, and P a point on the surface of the sphere of activity of the sun's attractive force, the bodies that pass through P may have all possible directions, but a direction making a small angle with PS is less probable than one making a greater angle. I cannot see that Prof. Schiaparelli has introduced this condition into his solution. Again, denoting the perihelion distance by D, the velocity by v, and the radius of the sphere ot the sun's activity by r, if we suppose all values of D equally probable between zero and D, Laplace finds (*Conn. des Tems* 1816, p. 216,) that the probabliity that the perihelion distance of the body will be comprised between zero and D is

$$1 - \frac{\sqrt{1 - \frac{D}{r}}}{r\,v} \cdot \sqrt{r^2 v^2 \left(1 + \frac{D}{r}\right) - 2\,D}; \, \ldots \ldots \quad (a)$$

He then multiplies this expression by $d\,v$, integrates between given limits, and dividing the integral by the greatest value of v finds the probability that v will be comprised within these limits. This division by the greatest value of v is omitted by Prof. Schiaparelli. There is another point in the solution given by Laplace that may be noticed. In order to perform the integration he changes the variable from v to z by means of the equation

$$r\,v = \frac{2\,D + z^2}{2\,z\sqrt{1 + \frac{D}{r}}};$$

and in finding the value of z for the limits of integration by solving this quadratic in z he chooses the negative sign before the irrational part of the root, and this choice ot the root throws the quantity that is afterwards made infinite into the denominator. It appears to me, therefore, that the solution given by Laplace is correct as considered from his own

standpoint. Preliminary to his solution of this problem, when speaking of the necessary near approach of comets to the sun in order that they may be visible, Laplace says: "One conceives that in order to approach so near the sun their velocity at the moment of their entrance into his sphere of activity must have a magnitude and direction comprised within narrow limits. It is required, therefore, to determine within these limits the ratio of the chances that give a sensible hyperbola to the chances that give an orbit which may be confounded with a parabola." But I cannot see that we gain much by the application of the calculus of probabilities to this question unless it be a somewhat clearer definition of the problem, since the conditions by which we restrict our solution are known beforehand, and as these conditions are varied our results will be changed. The condition of the question is this: We now have more than two hundred well known cometary orbits without a single example of a decided hyperbolic motion. No matter, therefore what may be the result of the application of the calculus of probabilities, it is quite certain that if comets enter our solar system they must do so under conditions that render very small the chance of hyperbolic orbits. On the other hand, we have a few, six or eight, orbits of meteors that show a hyperbolic motion. But in this case we have orbits that are determined from observations made with difficulty and which may be affected by large errors in the time of the flight of the meteor and in its position. The true course appears to be that our methods of observing meteors should be very much improved, so that we may obtain accurate observrtions of the duration of their flight and the position of their apparent paths. Then, having secure data, we can derive certain conclusions.

NOTE.—It may be interesting to see how Laplace gets the expression (a), given above. In addition to the notation already introduced, let a and e be the semi-axis major and the excentricity of the orbit; and let ω be the direction which the comet's motion makes with PS. Then we shall have the well known equations

$$v^2 = \frac{2}{r} - \frac{1}{a},$$

$$r\,v\,.\sin\omega = \sqrt{a(1-e^2)},$$

$$D = a\,(1-e);$$

Eliminating a and e from these equations, we have,

$$\sin^2 \omega = \frac{2\,D - \dfrac{2\,D^2}{r} + D^2\,v^2}{r^2\,v^2}:$$

Imagine a small sphere whose center is at the point P. The motion of the comet may be directed toward any point on the surface of the half of this sphere comprised within the sphere of the sun's activity. The probability of a direction forming the angle ω with PS, the radius-vector, will be $2\pi.\sin\omega$. Hence, dividing the integral $2\pi.\int\sin\omega\,.\,d\,\omega$ by the surface of the semi-sphere we shall have the probability that the direction of the motion will be comprised within the limits zero and ω. This probability is $1 - \cos\omega$; and from the above value of $\sin^2\omega$ we easily find the expression (a).

REPETENDS.

BY PROF. M. C. STEVENS, SALEM, OHIO.

(*Continued from page* 15.)

Let us illustrate by an example:

Suppose $\frac{1}{17}$ is to be reduced; we know from the *second principle* that the last figure must be 7, and from (1) we get $r_2 = 12$. It is convenient to write $r_2\,q_1$, $r_2\,q_2 + m_1$, &c., in a column, thus:

				$q_1 = 7$
$r_2\,q_1$	$= 84$	$\therefore$	$m_1 = 8$ and	$q_2 = 4$
$r_2\,q_2 + m_1$	$= 56$	$\therefore$	$m_2 = 5$ and	$q_3 = 6$
$r_2\,q_3 + m_2$	$= 97$	$\therefore$	$m_3 = 7$ and	$q_4 = 7$
$r_2\,q_4 + m_3$	$= 91$	$\therefore$	$m_4 = 9$ and	$q_5 = 1$
$r_2\,q_5 + m_4$	$= 21$	$\therefore$	$m_5 = 2$ and	$q_6 = 1$
$r_2\,q_6 + m_5$	$= 14$	$\therefore$	$m_6 = 1$ and	$q_7 = 4$
$r_2\,q_7 + m_6$	$= 49$	$\therefore$	$m_7 = 4$ and	$q_8 = 9$
$r_2\,q_8 + m_7$	$= 112$	$\therefore$	$m_8 = 11$ and	$q_9 = 2$
$r_2\,q_9 + m_8$	$= 35$	$\therefore$	$m_9 = 3$ and	$q_{10} = 5$
$r_2\,q_{10} + m_9$	$= 63$	$\therefore$	$m_{10} = 6$ and	$q_{11} = 3$
$r_2\,q_{11} + m_{10}$	$= 42$	$\therefore$	$m_{11} = 4$ and	$q_{12} = 2$
$r_2\,q_{12} + m_{11}$	$= 28$	$\therefore$	$m_{12} = 2$ and	$q_{13} = 8$
$r_2\,q_{13} + m_{12}$	$= 98$	$\therefore$	$m_{13} = 9$ and	$q_{14} = 8$
$r_2\,q_{14} + m_{13}$	$= 105$	$\therefore$	$m_{14} = 10$ and	$q_{15} = 5$
$r^2\,q_{15} + m_{14}$	$= 70$	$\therefore$	$m_{15} = 7$ and	$q_{16} = 0$

Here we stop, because the next figure will be 7, and it will commence to repeat.

We have, therefore, $\frac{1}{17} = .0588235294117647\frac{1}{17}$.

If the fraction had been $\frac{1}{13}$, we should have had

$$q_1 = 3, r_2 = 4, r_2\, q_1 = 12, r_2\, q_2 + m_1 = 9, r_2\, q_3 + m_2 = 36,$$
$$r_2\, q_4 + m_3 = 27, r^2\, q_5 + m_4 = 30 \;\therefore\; \tfrac{1}{13} = .076923\tfrac{1}{13}.$$

The operation may be very much abbreviated by taking advantage of the well known property of repetends, that after one half of the figures are obtained the second half may be found by subtracting each figure of the first half successively from 9.

We know when one-half of the digits are found because the remainder is then $d - 1$; and we know this in the above method from the fact that any number in the *column* divided by q_1 gives the corresponding remainder.

This property I demonstrate as follows:

Let $\frac{1}{d}$ be a fraction which reduces to a repetend, [verse order,
Let $q_1, q_2, q_3, \ldots q_n$ represent the successive digits of repetend in re-
Let $r_1, r_2, r_3, \ldots r_n$ " " " remainders in reverse order,
Let $t_1, t_2, t_3, \ldots t_n$ " " " terms of column,
Let $m_1, m_2, m_3 \ldots$ " " " tens " "

The column evidently stands thus:

$$t_1 = q_1$$

$$t_2 = r_2\, q_1 = 10\, m_1 + q_2 \;\therefore\; m_1 = \frac{q_1\, r_2 - q_2}{10}$$

$$_3 = r_2\, q_2 + m_1 = 10\, m_2 + q_3$$
$$\therefore m_2 = \frac{(10\, q_2 + q_1)r_2 - 10\, q_3 - q_2}{10^2}$$

$$t_4 = r_2\, q_3 + m_2 = 10\, m_3 + q_4$$
$$\therefore m_3 = \frac{(10^2\, q_3 + 10\, q_2 + q_1)r_2 - 10^2\, q_4 - 10\, q_3 - q_2}{10^3}$$

$$t_5 = r_2\, q_4 + m_3 = 10\, m_4 + q_5 \;\therefore\; m_4$$
$$= \frac{(10^3\, q_4 + 10^2\, q_3 + 10\, q_2 + q_1)r_2 - 10^3\, q_5 - 10^2\, q_4 - 10\, q_3 - q_2}{10^4}$$

&c., &c.

Substituting in t_3 for m_1 its value from t_2, in t_4 for m_2 its value from t_3, &c., we get

$$t_1 = q_1$$

$$t_2 = r_2\, q_1$$

$$t_3 = \frac{(10\ q_2 + q_1)r_2 - q_2}{10}$$

$$t_4 = \frac{(10^2\ q_3 + 10\ q_2 + q_1)r_2 - (10\ q_3 + q_2)}{10^2}$$

$$t_5 = \frac{(10^3 q_4 + 10^2 q_3 + 10 q_2 + q_1)r_2 - (10^2\ q_4 + 10\ q_3 + q_2)}{10^3}.$$

The law is now apparent. and we can write for the nth term

$$t_n = \frac{(10^{n-2} q_{n-1} + 10^{n-3} q_{n-2} + \ldots 10 q_2 + q_1)r_2 - (10^{n-3} q_{n-1} + \ldots 10 q_3 + q_2)}{10^{n-2}}.$$

Substituting a for $10^{n-3} q_{n-1} + 10^{n-4} q_{n-2} + \ldots 10 q_3 + q_2$, we get

$$t_n = \frac{(10\ a + q_1)r_2 - a}{10^{n-2}}. \quad \text{But } r_2 = \frac{d\ q_1 + 1}{10};$$

whence we readily get

$$t_n = \frac{[(10\ a + q_1)d + 1]q_1}{10^{n-1}} \quad \ldots\ldots \quad \text{(A)}$$

Let us next find the values of the remainders:

$$r_1 = 1,$$

$$r_2 = \frac{d\ q_1 + 1}{10},$$

$$r_3 = \frac{d\ q_2 + r_2}{10} = \frac{(10\ q_2 + q)d + 1}{10^2},$$

$$r_4 = \frac{d\ q_3 + r_3}{10} = \frac{(10^2\ q_3 + 10\ q_2 + q_1)d + 1}{10^3},$$

$$\vdots$$

$$r_n = \frac{(10^{n-2} q_{n-1} + 10n^{n-3} q_{n-2} + \ldots . 10 q_2 + q_1)d + 1}{10^{n-1}} = \frac{(10a + q_1)d + 1}{10^{n-1}}. \text{(B)}$$

Dividing (A) by (B), member by member, we get

$$\frac{t_n}{r_n} = q_1 \text{ or } r_n = \frac{t_n}{q^1}. \quad \text{Q. E D.}$$

I also demonstrate the property above referred to, viz: That after one-half of the digits are found the second half may be obtained by subtracting each figure already obtained successively from 9, as follows:

Let $\frac{1}{d}$ be a fraction which when reduced to a decimal gives a repetend in which the remainder $d-1$ occurs We evidently have

$$\frac{1}{d} = .R\,\frac{d-1}{d}, \text{ and } \frac{d-1}{d} = .R'\,\frac{1}{d};$$

in which R contains the digits before reaching the remainder $d-1$, and R' the digits after reaching that remainder. But

$$\frac{1}{d} + \frac{d-1}{d} = 1;$$

hence the sum of the second members of the equation must also equal 1. R' must contain the same number of places as R, since it contains the same sequence of figures differently arranged. Now since the fractional parts when added give 1, $R + R' = 99\,.\,.\,.\,.\,.\,9$ repeated as many times as there are digits in $R + R'$, whence $R = 99\ldots 9 - R'$. Q. E. D.

The *second principle* may be proved as follows: As has been shown, the remainder next to the last is

$$r_2 = \frac{d\,q_1 + 1}{10}.$$

In order that this may be an integer the *units* of $d\,q_1$ must be 9; and it is evident that if the units of d be 1, then q_1 must be 9; if it be 3 then q_1 must be 3; if it be 7 then q_1 must be 7; but if it be 9 then q_1 must be 1.

In conclusion I will remark that it is improper to call repetends, as they are usually represented, *decimals*. For an example. $.\dot{3}$ is not a decimal, but should be read *three-ninths;* so $.\dot{1}\dot{6}$ should be read *sixteen ninety-ninths;* $.1\dot{4}$ should be read $1\frac{4}{9}$ *tenths*, &c.

SOLUTION OF A PROBLEM.

BY G. W. HILL, ESQ., NYACK TURNPIKE, N. Y.

The following problem appeared in the *Mathematical Monthly*, (Vol. 1, p. 29), and no solution was published in that periodical:

"Show that the product of six entire consecutive numbers cannot be the square of a commensurable number."

Since the square root of every integer, not an exact square, is a surd, it will be sufficient to show that the product cannot be the square of an integer. Let the six numbers be denoted, n being an odd integer, by

$$\frac{n-5}{2},\ \frac{n-3}{2},\ \frac{n-1}{2},\ \frac{n+1}{2},\ \frac{n+3}{2},\ \frac{n+5}{2}.$$

Then it is required to prove the impossibility of

$$\frac{n^2-25}{4}\cdot\frac{n^2-9}{4}\cdot\frac{n^2-1}{4}=\square.$$

Let us put

$$\frac{n^2-9}{4}=x,$$

where x is integral since it is the product of two integers. Then it will suffice to prove the impossibility of

$$x(x+2)(x-4)=\square.$$

Let us suppose $x=k^2\,y$, where k^2 is the largest square factor contained in x, and thus y will be divisible by no square other than unity. Then we have to prove the impossibility of

$$y(k^2\,y+2)(k^2y-4)=\square.$$

But since y contains no square factor, both members of this equation must be divisible by y^2; this demands that $2\times 4=8$ be divisible by y. Hence, having regard to the restriction on the form of y, if the equation is possible, it can be so only for the values $y=1$ or $y=2$. The first gives

$$(k^2-1)^2-\square=9,$$

which is satisfied only by $k^2-1=5$, or $k=\sqrt{6}$, a surd, thefore y cannot be unity. For $y=2$, we have

$$2(k^2+1)(k^2-2)=\square.$$

But every square is of the form $3n$ or $3n+1$; if these are substituted in succession for k^2 in the left-hand member of the last equation, it will be seen that the resulting quantities are of the form $3n+2$, and thus cannot be squares. Therefore y cannot be 2, and the impossibility is completely demonstrated.

Evidently the proposition might be ennunciated in the much more general manner:—

The product of any number of consecutive integers cannot be an exact power of any degree.

Prof. J. E. Oliver, Cornell University, Ithaca, N. Y. writes:

Your correspondent's "query" is answered by some standard writer on Physics (probably Dagain or Ganot, or both,)—and the picture there is better than I can draw.

The stream, adhering to the ball, is deflected toward it. If the ball is to the right of the stream's axis more water passes to the ball's left and is deflected to the right than the reverse; $\therefore$ the stream is, on the whole, deflected to the right, $\therefore$ the ball is drawn to the left, since action and reaction are opposite.

Note on the Ellipse.—From the well known property of the ellipse, that the rectangle of the abscissas is to the square of the ordinate as the square of the major axis is to the square of the minor axis, we get the equation

$$y = \frac{b}{a}\sqrt{a^2 - x^2},$$

in which a and b are the semi-axes and x and y the co-ordinates; the origin being at the center. If we put

$$x = \frac{2\,a\,n}{n^2+1} \text{ or } \frac{a(n^2-1)}{n^2+1},$$

y will always be rational and equal to

$$b\,.\,\frac{n^2-1}{n^2+1} \text{ or } b\,.\,\frac{2\,n}{n^2+1}.$$

Assume $n^2 + 1 = a$, then will $2n$ and $n^2 - 1$ represent the abscissas corresponding to the ordinates

$$b\,.\,\frac{n^2-1}{n^2+1} \text{ and } b\,.\,\frac{2\,n}{n^2+1},$$

where n may be any number whatever greater than *one*.

Hence, if we divide the semi-major axis of any ellipse into $n^2 + 1$ equal parts we may find two points in the axis whose distance from the center or from the extremity of the axis may be expressed in integral parts of n, and each of which has a rational ordinate.

Example.—Put $n = 2$, then is the semi-major axis $= n^2 + 1 = 5$, and the two points in the axis are respectively $2\,n$ and $n^2 - 1$, or 4 and 3 dis-

tant from the center, or 1 and 2 distant from the extremity of the axis, and their corresponding ordinates are $\frac{3}{5}b$ and $\frac{4}{5}b$.

If $n = 3$, we have $n^2 + 1 = 10$, and the distances of the two points from the extremity of the axis are 4 and 2, and their corresponding ordinates are $\frac{4}{5}b$ and $\frac{3}{5}b$. If $n = 4$, $n^2 + 1 = 17$, and the distances of the two points from the extremity of the axis are 9 and 2, and their corresponding ordinates are $\frac{15}{17}b$ and $\frac{8}{17}b$. By assigning other values to n other rational ordinates may be found to any extent that may be desired.

It is remarkable that whatever value may be assigned to n, one of the points in the axis that has a rational ordinate will always be at a distance from the extremity of the axis equal to two of the units in n.

Problem.—To integrate $\dfrac{dx}{1 + e\cos x}$.

$$\text{We have } \frac{dx}{1 + e\cos x} = \frac{dx}{1 + e\cos^2 \frac{1}{2}x - e\sin^2 \frac{1}{2}x}.$$

Put tan. $\frac{1}{2}x = z$; then $\frac{1}{2}\sec^2 \frac{1}{2}x\, dx = dz$, and $\sec^2 \frac{1}{2}x = 2\, dz$.

Therefore the required integral

$$= \int \frac{\dfrac{dx}{\cos^2 \frac{1}{2}x}}{\dfrac{1}{\cos^2 \frac{1}{2}x} + e - e\tan^2 \frac{1}{2}x} = \int \frac{\sec^2 \frac{1}{2}x\, dx}{\sec^2 \frac{1}{2}x + e - e\tan^2 \frac{1}{2}x}$$

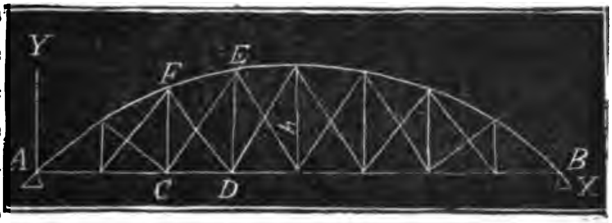
$$= \int \frac{2\, dz}{1 + z^2 + e - e z^2} = \frac{2}{1 + e}\int \frac{dz}{1 + \dfrac{1 - e}{1 + e}z^2}$$

$$= \frac{2}{1 + e}\sqrt{\frac{1 + e}{1 - e}}\int \frac{\sqrt{\dfrac{1 - e}{1 + e}}\, dz}{1 + \dfrac{1 - e}{1 + e}z^2}$$

$$= \frac{2}{\sqrt{1 - e^2}}\tan^{-1}\sqrt{\frac{1 - e}{1 + e}}\, z = \frac{2}{\sqrt{1 - e^2}}\tan^{-1}\sqrt{\frac{1 - e}{1 + e}}\tan \tfrac{1}{2}x + C.$$

—James E. Clark, Professor of Mathematics, *William Jewell College*, Liberty, Mo.

PROBLEM.—BY S. W. SALMON, MOUNT OLIVE, N. J.—To find the greatest strain on the braces and counter-braces of a truss, produced by the passage over it of a load of uniform weight (q) per unit ot length, and of length (l) equal to the length of the truss.



Let the load pass from A to B along the lower chord. Let $S =$ the strain on any brace produced by the passage of the load; H, the horizontal component of the force producing the strain; and ϕ the angle which the brace makes with the horizontal. Then

$$S = \frac{H}{\cos\phi}.$$

Take any panel $F\,C\,D\,E$. Let $A\,C = a$, $A\,D = b$, $CF = y'$, $DE = y''$, $H' =$ horizontal force at F, and $H'' =$ horizontal force at E. Then $H = H' - H''$. When a part of the load of length x has passed on the chord, the reaction of the abutment at B is

$$\frac{q\,x^2}{2\,l}.$$

Three different expressions may be obtained for H. 1st, when $x<a$; 2d, when $x>a$ and $<b$, and 3d. when $x>b$. The 1st and 3d expressions have no *maximum* value. In the 2d case, by taking the moments around D we find

$$H'' = \frac{q\,(l-b)x^2}{2\,l\,y''};$$

taking the moments around C, we find

$$H' = \frac{q}{y'}\left(a\,x - \frac{a\,x^2}{2\,l} - \frac{a^2}{2}\right).$$

Hence $H = H' - H''$

$$= \frac{q}{y'}\left(a\,x - \frac{a\,x^2}{2\,l} - \frac{a^2}{2}\right) - \frac{q(l-b)x^2}{2\,l\,y''}. \quad . \quad . \quad . \quad . \quad (1)$$

$$\text{Put } u = \frac{1}{y'}\left(a\,x - \frac{a\,x^2}{2\,l} - \frac{a^2}{2}\right) - \frac{(l-b)x^2}{2\,l\,y''} = \text{a maximum.}$$

Differentiating, $\frac{du}{ax} = \frac{a}{y'} - \frac{a\,x}{l\,y'} - \frac{(l-b)x}{l\,y''} = 0.$

$$\therefore x = \frac{a\,l\,y''}{a\,y'' + (l-b)y'};$$

$$\frac{d^2u}{dx^2} = -\frac{a}{l\,y'} - \frac{l-b}{l\,y''},$$

which, being negative, indicates a maximum.

Substituting the above value of x in (1) we find the max. value of H

$$= \frac{q\,a^2}{2}\left(\frac{(l-a)y'' - (l-b)y'}{a\,y'\,y'' + (l-b)y'^2}\right) \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$\therefore$ the max. value of S

$$= \frac{q\,a^2}{2\cos\psi}\left(\frac{(l-a)y'' - (l-b)y'}{a\,y'\,y'' + (l-b)y'^2}\right).$$

If the top chord be straight, $y' = y'' = h$, and

$$S = \frac{q\,a^2}{2\,h\cos\psi}\left(\frac{b-a}{l-(b-a)}\right).$$

Let y' and y'' be ordinates of a parabola whose axis is vertical. The equation of a parabola referred to rectangular co-ordinates, A being the origin, and h the ordinate of its highest point, is

$$y = \frac{4h}{l^2}(l\,x - x^2) \therefore y' = \frac{4h}{l^2}(l\,a - a^2) \text{ and } y'' = \frac{4h}{l^2}(l\,b - b^2).$$

Substituting these values of y' and y'' in (2)

$$H = \frac{q\,l^2}{8h}\left(\frac{b-a}{l+b-a}\right)$$

which is constant if $(b-a)$ be constant, that is, if the panels be of equal length.

$$\therefore S = \frac{q\,l^2}{8\,h\cos\psi}\left(\frac{l-a}{l+b-a}\right).$$

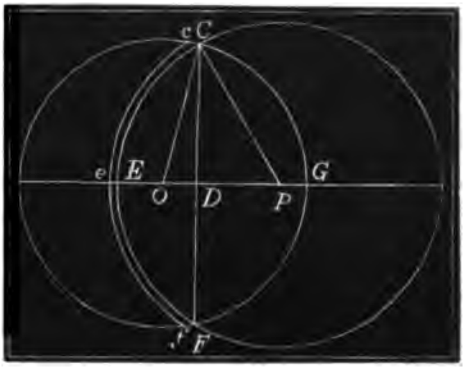
If the truss sustains also a load uniformly distributed along the lower chord, then, when the upper chord is straight,

$$S = \frac{q\,a^2}{2\,h\cos\psi}\left(\frac{b-a}{l-b+a}\right) + \frac{p}{2\,h\cos\psi}[(l\,b - b^2) - (l\,a - a^2)]$$

where $p =$ the weight per unit of length of the uniform load.

TO FIND THE AREA COMMON TO TWO INTERSECTING CIRCLES.

BY ARTEMAS MARTIN, MATHEMATICAL EDITOR SCHOOLDAY MAGAZINE.



Let O and P be the centers of two intersecting circles. Put $OP = a$, $OC = r$, and suppose the radius CP variable and $= x$. The circles will intersect if x is not greater than $r + a$ nor less than $r - a$. With center P and radius $x + dx$ describe the arc cef indefinitely near CEF. Then

$$DP = \frac{x^2 + a^2 - r^2}{2a}, \text{ arc } CEF = 2x \cos^{-1}\left(\frac{x^2 + a^2 - r^2}{2ax}\right)$$

and the differential of the area $CEFG$ is

$$2x \cos^{-1}\left(\frac{x^2 + a^2 - r^2}{2ax}\right) dx.$$

Putting $\triangle$ for the area sought, we have

$$\triangle = \int_{r-a}^{x} 2x \cos^{-1}\left(\frac{x^2 + a^2 - r^2}{2ax}\right) dx.$$

$$\int 2x \cos^{-1}\left(\frac{x^2 + a^2 - r^2}{2ax}\right) dx = x^2 \cos^{-1}\left(\frac{x^2 + a^2 - r^2}{2ax}\right)$$

$$+ \int \frac{x(r^2 - a^2 + x^2)dx}{\sqrt{4a^2r^2 - (r^2 + a^2 - x^2)^2}}.$$

Put $r^2 + a^2 - x^2 = y$, then

$$\int \frac{x(r^2 - a^2 + x^2)dx}{\sqrt{4a^2r^2 - (r^2 + a^2 - x^2)^2}} = \int \frac{-\frac{1}{2}(2r^2 - y)dy}{\sqrt{4a^2r^2 - y^2}}$$

$$= \int \frac{-r^2 dy}{\sqrt{4a^2r^2 - y^2}} + \int \frac{\frac{1}{2}y\,dy}{\sqrt{4a^2r^2 - y^2}} = r^2 \cos^{-1}\left(\frac{y}{4ar}\right) - \tfrac{1}{2}\sqrt{4a^2r^2 - y^2}.$$

$$\therefore \triangle = x^2 \cos^{-1}\left(\frac{x^2+a^2-r^2}{2\,a\,x}\right) + r^2 \cos^{-1}\left(\frac{a^2+r^2-x^2}{2\,a\,r}\right) - a\sqrt{r^2-\left(\frac{a^2+r^2-x^2}{2\,a}\right)^2}.$$

When $x = R$,

$$\triangle = R^2 \cos^{-1}\left(\frac{a^2-r^2+R^2}{2\,a\,R}\right) + r^2 \cos^{-1}\left(\frac{a^2+r^2-R^2}{2\,a\,r}\right) - a\sqrt{r^2-\left(\frac{a^2+r^2-R^2}{2\,a}\right)^2},$$

which agrees with the result obtained by the ordinary method.

The above formula may be readily adapted to any special case.

When the center P is on the circumference of the other circle, $a = r$, and

$$\triangle = R^2 \cos^{-1}\left(\frac{R}{2\,r}\right) + 2\,r^2 \sin^{-1}\left(\frac{R}{2\,r}\right) - \tfrac{1}{2}R\sqrt{4\,r^2 - R^2}.$$

If the circles are equal, $R = r$ and

$$\triangle = 2\,r^2 \cos^{-1}\left(\frac{a}{2r}\right) - \tfrac{1}{2}a\sqrt{4\,r^2 - a^2}.$$

When they are equal and the center of one on the circumference of the other, $R = r$, $a = r$ and

$$\triangle = r^2\left(\frac{2}{3}\pi - \frac{1}{2}\sqrt{3}\right).$$

PROBLEMS.

5. In a plane triangle there are given the three lines bisecting the angles, a, b and c, to find the sides.—Communicated by Dr. David S. Hart, Stonington. Conn.

6. Find a convenient formula for calculating the capacity of a cistern constructed as follows, viz: Having a lower concavity which is a spherical segment whose versed sine is a and chord $2\,r$, a central cylindrical part whose radius is r and perpendicular height h, and an upper concavity which is a spherical segment whose versed sine is b and chord $2\,r$.—Communicated by Frank Pelton, C. E., Des Moines, Iowa.

7. Multiply $\sqrt[3]{1+\sqrt[3]{1+\sqrt[3]{1+\&c.}}}$ by $\sqrt{1+\dfrac{1}{\sqrt{1+\dfrac{1}{\sqrt{1+\&c.}}}}}$

and express the product in a finite number of terms.—Communicated by Prof. Daniel Kirkwood, Bloomington, Ind.

8. A ball *rolls* down the convex surface of a fixed sphere, the friction* being just sufficient to prevent sliding; find the point where it leaves the sphere.—Communicated by ARTEMAS MARTIN, Mathematical Editor of *Schoolday Magazine*, Erie, Pa.

9. P and Q denoting two entire functions of x, such that we have $\sqrt{1-P^2} = Q\sqrt{1-x^2}$, we have necessarily

$$\frac{dP}{\sqrt{1-P^2}} = n\frac{dx}{\sqrt{1-x^2}};$$

n denoting an entire number.—Communicated by PROF. HALL, Washington, D. C.

10. The chief justice of a court makes a large number of legal decisions. Afterward it is found that 50 per cent. of these decisions are erroneous. Required to determine the legal knowledge of the judge.—Communicated by PROF. HALL.

EDITORIAL NOTES.

In presenting this, the second number, to our readers we hope it will be found to be something of an improvement on the first. We do not intend to use up our space in apologizing for past delinquencies, or in promises of future improvement, but we hope to learn from experience, and profit by the advice and criticisms of our correspondents.

A very considerable number of typographical errors occur in our first number. Some of these we noticed, but too late to remove them, and others have been pointed out by our correspondents. As all the errors that have been noticed are unimportant and cannot mislead any of our readers, we will allow each reader to correct them for himself; and we will here remark that, in the future, though we will use as much care as we can command to avoid typographical errors, and will, when *material*, if noticed or pointed out to us, subsequently insert the correction; but all immaterial errors, that is, all such as cannot mislead the reader, we will pass in silence; though we will be thankful to have our attention called to them, so that we can guard against similar errors in

*If in this question the proper substitutions be made from the equations of motion as given in any standard work on Mechanics, Bartlett's for instance, it is found that the ball will quit the sphere at the moment when it has descended through seven-seventeenths of the radius of the sphere, supposing it to have started from the summit. But on the supposition that it *slides* down the sphere without friction, then it would quit the sphere when it has descended through one-third the radius.

Prof. Peck says, however, that "If a body be placed on an inclined plane and abandoned to the action of its own weight it will either slide or roll down the plane, provided there be no friction between it and the plane. If the body is spherical it will roll, and in this case the friction may be disregarded . . . . and equations (67) and (68) will be immediately applicable." [See Peck's Mechanics, p. 158.]

Will correspondents explain this apparent discrepancy.—ED.

future issues. The delinquency, in all such cases, must be charged by correspondents to the printers and proof reader.

Some criticisms have been presented relative to the subject matter of the first number. Some of the articles are regarded by the critics as not possessing sufficient interest to entitle them to a place in such a journal. In answer, and without particularizing, we remark, that we do not entertain the same opinion, and we do not feel it a duty to apologize for *any* of the articles published. On the contrary, we believe that if the mind is allowed to dwell for a sufficient time upon almost any subject it will be found to be interesting. It is true that when any subject is thoroughly understood and perfectly familiar to us it ceases to possess much interest, as the mind is, and ought to be, intent upon *new* conquests. Hence, it would be vain to hope that all the readers would be interested with *every* article that should appear in *any* periodical, and especially in a periodical devoted to mathematics, which, to offer any chance of success, must occasionally present subjects that will tax the capacity of its ablest readers, and also subjects that are comprehensible to the less advanced reader.

Acting upon these convictions it is our desire and will be our aim to present something that will be adapted to the tastes and desires of each and every one of our readers, but we have no expectation of interesting *all* with everything that may be presented. We add, as a further explanation of the course we expect to pursue, that, when persons of known and acknowledged ability as mathematicians, present articles for publication, we will, as we find room, give them a place in our pages, even though we should not, from the limited examination that we may be able to give them, discover any special merit in the article. We will assume that it merits insertion from its presentation by a responsible author. But, on the other hand, to guard our pages from improper matter, when the author is not known to us as a mathematician of acknowledged capacity, the criterion must be our appreciation of the *merits* of his article.

We do not feel called upon to apologize for the somewhat extended article with which we introduce the present number. Though we have added four pages to this number in consequence of the space occupied by the first article, we have done so, not because we thought the number would possess less than the average interest without such addition, but because we are aware that a very considerable number of our readers feel little interest in astronomical questions of the kind there discussed, and might feel somewhat disappointed by a want of sufficient *variety* in the number.

We embrace this opportunity to say, that to *us* the consideration of comets and meteors is one of the most interesting subjects in physical astronomy; and that it appears to be a necessary conclusion from Prof. Hall's discussion of the subject—assuming the nebular hypothesis to be true in its main points—that the comets, which appear at intervals in our skies are not wanderers from other stars, but that most of them, if not all, belong to our system, and, probably, originated near the verge of the system. For, when the nebulous matter from which the system has been condensed extended to the neutral point of attraction between this and a contiguous nebula or star, the attracting influence at the out-skirts of the nebula must have been but feeble, so that as condensation progressed out-lying clouds of gas would have been likely to remain behind near the point of neutral attraction, and these out-lying clouds would, in time, yield to the attracting force of the nearest nebula or star, and approach it as a comet or meteor in a parabolic or elliptic orbit. And although a few meteors have been observed that seemed to have too great a velocity to allow their motion to be referred to parabolas, yet if we admit they came from the stellar regions the difficulty of accounting for the initial velocity with which they must have entered our system is as great, it would seem, as to account for that velocity within our system; and when we take into consideration the uncertainty attending the observations, as indicated by Prof. Hall, the doubt as to their introduction from without is increased.

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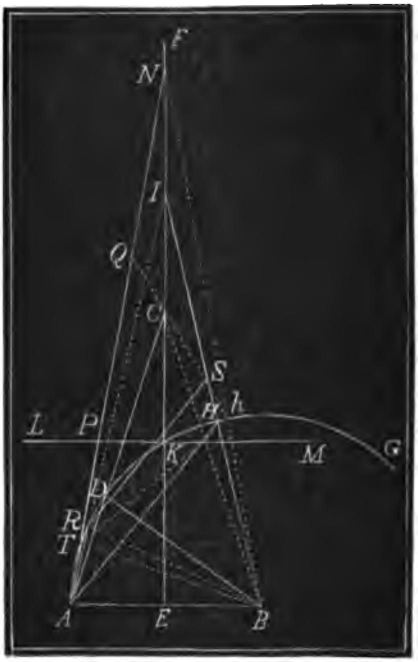
PROPERTIES OF POLYGONS.

BY ELIAS SCHNEIDER, A. M., SUNBURY, PA.

In the following article I propose to point out some interesting relations that exist between certain lines and areas of polygons, that, so far as I know, have never heretofore been announced; and to present a method for the construction of certain polygons that have never been constructed geometrically, which, though not strictly geometrical, yet from the analogy of the construction to strictly geometrical constructions I am induced to believe that geometrical constructions for these polygons may yet be found.



1. Draw the line $A\ B =$ the unit of our scale, and bisect it in E, and at E erect the indefinite perpendicular $E\ F$. With $A\ B$ as radius describe the arc $A\ G$, and draw the straight line $A\ C$ cutting the arc $A\ G$ in D, so that $D\ C$ shall equal $A\ B$. Then is $A\ B$ one side of a *pentagon* inscribed in a circle which passes through the three points A, B and C.

2. Draw the straight line $B\ I$ cutting the arc $A\ G$ in H so that $H\ I = A\ H$. Then is $A\ B$ one side of a *heptagon* inscribed in a circle which passes through the three points A, B and I.

3. Through K, the point where the arc $A\ G$ cuts the perpendicular $E\ F$, draw the line $L\ M$ parallel to $A\ B$, and draw the straight line $A\ N$ cutting $L\ M$ in P so that $P\ N$ shall equal twice $A\ B$. Then is $A\ B$ one

side of a *nonagon* inscribed in a circle which passes through the three points A, B and N.

The demonstration of these constructions is easy and is theretore omitted for the sake of brevity.

It in the heptagon whose sides each equals *one*, we put the chord of the arc which contains three of the equal sides $= 2 + x$, then will $2 + x$ be the radius of a circle in which if a polygon of double the number of sides (14) be described each of these fourteen equal sides will $=$ *one*; and the length of one side of a heptagon described in a circle whose radius is one will be $\sqrt{1 - x}$.

Also, if in the nonagon each of whose sides equals *one* we put the chord of the arc which contains three of the equal sides $= 2 + x$, then will one side of the nonagon described in a circle whose radius is one be $\sqrt{1 - x}$. And in general, if in a polygon of n equal sides (n being any number greater than six) each of which equals *one*, we put $2 + x =$ the chord of the arc which contains three of the equal sides, $\sqrt{1 - x}$ will be the length of one side of a polygon of n sides described in a circle whose radius is *one*.

Let β represent the angle subtended by one of the equal sides of a polygon of n sides. Then is

$$x = 2\cos\beta - 1. \quad \therefore \sqrt{1 - x} = \sqrt{2 - 2\cos\beta}$$

$$= \sqrt{2}\sqrt{1 - \cos\beta} = \sqrt{2}\sqrt{1 - \sqrt{1 - \sin^2\beta}} = \sqrt{1 + \sin\beta} - \sqrt{1 - \sin\beta}.$$

That is, one side of any regular polygon inscribed in a circle whose radius is *one*, is represented in functions of the sine of the arc subtended by that side, by

$$\sqrt{1 + \sin\beta} - \sqrt{1 - \sin\beta},$$

or by its development

$$\sin\beta + \tfrac{1}{8}\sin^3\beta + \tfrac{25}{512}\sin^5\beta + \&c. \quad . \quad . \quad . \quad . \quad . \; (1)$$

By assigning any value to β, n will be determined. And if β be taken very small, n times (1) will represent approximately the circumference of a circle whose radius is one.

A very interesting relation exists between the isosceles triangles which are formed in the construction of polygons.

The triangle $A\ B\ C$ of the pentagon is divided into two triangles by the line $B\ D$. Multiply the area of the triangle $A\ B\ D$ by the line $B\ C$ and the product equals the area of the triangle $B\ D\ C$. Multiply the area of

this triangle by the line $B\ C$ and the product equals the area of the triangle $A\ B\ C$. That is, these three triangles form a geometrical series of which $B\ C$ is the ratio. The triangle $B\ D\ C$ is a mean proportional between the other two.

The triangle $A\ B\ I$ of the heptagon divides itself into three isosceles triangles; but only two of these together with the whole triangle form a geometrical series. Multiply the area of the triangle $A\ B\ R$ by $B\ I$ and the product will be equal to the area of the triangle $B\ R\ S$, and this last triangle multiplied by $B\ I$ will give the area of the triangle $A\ B\ I$. But the area of $A\ B\ R$ multiplied by $R\ I$ will give the area of $R\ S\ I$, and $R\ S\ I$ multiplied by $B\ S$ will give the area of $B\ R\ S$. Therefore, since $A\ B\ R \times R\ I \times B\ S = B\ R\ S$, $R\ I \times B\ S = B\ I$. And because $B\ H = A\ B$, therefore the triangle $A\ B\ H$ equals the triangle $B\ R\ S$.

The triangle $A\ B\ N$ of the nonagon divides itself into four isosceles triangles, but only two of these together with the whole triangle form a geometrical series.

Multiply the area $A\ B\ T$ by $B\ N$ and the product will be equal to the area $T\ h\ Q$, and this last multiplied by $B\ N$ will give the area of $A\ B\ N$. But the area of the triangle $A\ B\ T$ multiplied by $h\ N$ gives the area of the triangle $N\ Q\ h$, and the area of $A\ B\ T$ multiplied by $N\ T$ gives the area of the triangle $B\ T\ h$.

If the triangle of the polygon of eleven sides be constructed, the triangle will divide itself into five isosceles triangles and the third from the base will be the mean proportional. But in the triangle of the polygon of thirteen sides the fourth triangle from the base will be the mean proportional, there being six triangles formed in this case.

If in the pentagon whose sides each $=$ *one* we put the chord which contains two of the equal sides $= 1 + x$, we get for the equation which determines the value of x,

$$x^2 + x = 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (a)$$

And if in the heptagon and nonagon, whose sides $=$ *one*, we put the chord which contains three of the equal sides $= 2 + x$, we get for the equations respectively, for the heptagon

$$x^3 + 4\,x^2 + 3\,x = 1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (b)$$

and for the nonagon

$$x^3 + 3\,x^2 = 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (c)$$

And in general, we have, for the equation of a polygon of n sides, in which $2 + x$ represents the chord which contains three of the equal sides, (each

of which $=$ *one*), r the radius of the circumscribing circle, and β the angle subtended by one of the equal sides,

$$x = \frac{r^2-1}{r^2}, \text{ and } r = \frac{1}{\sqrt{2\,(1-\cos\beta)}}.$$

SOLUTION OF TWO INDETERMINATE PROBLEMS.

BY GEO. R. PERKINS, LL. D., UTICA, N. Y.

PROBLEM I.—Find n numbers in arithmetical progression, such that the sum of their cubes shall be a square number.

Assume the n terms of the progression as follows:

$$x-\frac{n-1}{2}\,d;\; x-\frac{n-3}{2}\,d;\; \ldots . x-\tfrac{1}{2}d;\; x+\tfrac{1}{2}d;\; \ldots . x+\frac{n-3}{2}\,d;\; x+\frac{n-1}{2}\,d,$$

which corresponds to the case when n is an even number, and

$$x-\frac{n-1}{2}\,d;\; x-\frac{n-3}{2}\,d;\; \ldots . x-d;\; x;\; x+d;\; \ldots . x+\frac{n-3}{2}\,d;\; x+\frac{n-1}{2}\,d,$$

which corresponds to the case when n is an odd number.

The sum of the cubes of the n terms, when n is even, is

$$n\,x^3 + \tfrac{3}{2}\,[1^2 + 3^2 + 5^2 + \ldots . (n-3)^2 + (n-1)^2]\,d^2\,x.$$

When n is odd, the sum of the cubes is

$$n\,x^3 + 6\,[1^2 + 2^2 + 3^2 + \ldots . (\tfrac{1}{2}\,\overline{n-3})^2 + (\tfrac{1}{2}\,\overline{n-1})^2]\,d^2\,x.$$

Each of these expressions, when simplified, becomes

$$n\,x\,[x^2 + (n^2-1)\,(\tfrac{1}{2}\,d)^2] \;.\;.\;.\;.\;.\;.\;.\;.\; (a)$$

and this must be a square number.

$$\text{Assume } x^2 + (n^2-1)\,(\tfrac{1}{2}\,d)^2 = 4\,n\,x\,t^2,$$

and the above will become $4\,n^2\,x^2\,t^2 =$ a square.

$$\text{Solving } x^2 + (n^2-1)\,(\tfrac{1}{2}\,d)^2 = 4\,n\,x\,t^2$$

for x, we find

$$x = 2\,n\,t^2 + \sqrt{4\,n^2\,t^4-(n^2-1)\,(\tfrac{1}{2}\,d)^2} \;.\;.\;.\;.\;.\; (b)$$

Hence, in order that this value of x may be rational, we must make

$$4\,n^2\,t^4 - (n^2-1)\,(\tfrac{1}{2}\,d)^2 = \text{a square} = [2\,n\,t^2 - s\,(\tfrac{1}{2}\,d)]^2$$

$$\text{This gives } d = \frac{8\,n\,s\,t^2}{s^2+n^2-1}, \text{ and then}$$

$$x = \frac{4\,n\,(n^2-1)\,t^2}{s^2+n^2-1}.$$

The first term $= x - \frac{n-1}{2} d = \frac{4n(n-1)(n+1-s)t^2}{s^2+n^2-1}$.

The last term $= x + \frac{n-1}{2} d = \frac{4n(n-1)(n+1+s)t^2}{s^2+n^2-1}$.

And the sum of the cubes of all the terms

$$= 4n^2 x^2 t^2 = \frac{64 n^4 (n^2-1)^2 t^6}{(s^2+n^2-1)^2}.$$

If we take $s=1$ and $t=\frac{1}{2} n r$, the foregoing expressions will be integral, as follows:

$$\left.\begin{array}{ll} & d = 2 n r^2 \\ \text{The first term} & = n^2 (n-1) r^2 \\ \text{The last term} & = n(n-1)(n+2) r^2 \\ \text{The sum of the cubes} & = n^6 (n^2-1)^2 r^6 \end{array}\right\} \ldots\ldots (c)$$

Again, if we wish our terms to correspond to the series of natural numbers, we assume, in (c), as follows:

$$r = \frac{1}{2m} \text{ and } n = 2m^2 = \text{number of terms,}$$

and we have

$$\left.\begin{array}{ll} & d = 1 \\ \text{The first term} & = m^2 (2m^2-1) \\ \text{The last term} & = (m^2+1)(2m^2-1) \\ \text{Sum of the cubes} & = m^6 (4m^4-1)^2 \end{array}\right\} \ldots\ldots (d)$$

As particular examples of formula (c), take $r = 1$ and $n =$ number of terms $= 3, 4$ and 5, in succession, and we find,

$$(18)^3 + (24)^3 + (30)^3 = (216)^2,$$
$$(48)^3 + (56)^3 + (64)^3 + (72)^3 = (960)^2,$$
$$(100)^3 + (110)^3 + (120)^3 + (130)^3 + (140)^3 = (3000)^2.$$

As particular examples of formula (d), take $m = 1, 2, 3$ and 4, in succession, and we find,

$$1^3 + 2^3 = 3^2,$$
$$(28)^3+(29)^3+(30)^3+(31)^3+(32)^3+(33)^3+(34)^3+(35)^3 = (504)^2,$$
$$(153)^3+(154)^3+ \ldots . (169)^3+(170)^3 = (8721)^2,$$
$$(496)^3+(497)^3+ \ldots . (526)^3+(527)^3 = (65472)^2.$$

REMARK.—A solution of this Problem, when $d = 1$, is given by M. Eugène Catalan in the XX.th volume of the Acts of the Accademia Pontificia Dè Nouvi Liucei, Rome, 1866.

This solution is very satisfactory, but very lengthy, involving very large numbers, and is, moreover, not a general solution.

His numbers are

$$[887240758600]^3 + [887240758601]^3 + \ldots . + [1041543499225]^3$$
$$= [967473261775 \times 771518703l3]^2.$$

Problem II.—Find n consecutive terms, in the natural series of numbers, such that the sum of their cubes shall be a cube number.

Using the same notation as in the last problem, and making $d = 1$, our expression (a) for the sum of the cubes becomes

$$n\,x\,[x^2 + \tfrac{1}{4}(n^2 - 1)],$$

and this must be a cube, which is obviously the case, when $x = \frac{1}{2}$.

We will therefore assume $x = \frac{1}{2} + r$, and our expression will become

$$n\left\{\frac{n^2}{8} + \frac{n^2+2}{4}r + \frac{3}{2}r^2 + r^3\right\}.$$

Put $n = p^3$ = number of terms, and this will become

$$p^3\left\{\frac{p^6}{8} + \frac{p^6+2}{4}r + \frac{3}{2}r^2 + r^3\right\} = \text{cube} = p^3\left\{\frac{p^2}{2} + r\right\}^3.$$

Hence $r = \dfrac{p^4 - 2p^2 - 2}{6}$, and $x = \frac{1}{2} + r = \dfrac{p^4 - 2p^2 + 1}{6}$.

$$\left.\begin{aligned} &\text{First term} = x - \frac{p^3-1}{2} = \frac{p^4-3p^3-2p^2+4}{6}, \\ &\text{Last term} = x + \frac{p^3-1}{2} = \frac{p^4+3p^3-2p^2-2}{6}, \\ &\text{The sum of the cubes} = \left\{\frac{p(p-1)(p+1)(p^2+2)}{6}\right\}^3. \end{aligned}\right\} \dots\dots (e)$$

As particular cases, first suppose $p = 2$ and we find

$$\frac{p^4 - 3p^3 - 2p^2 + 4}{6} = -2, \text{ the first term.}$$

Hence the 8 terms denoted by p^3 will be $-2, -1, 0, 1, 2, 3, 4, 5$.

The cubes of the two negative values -1 and -2 will balance the cubes of the two corresponding positive values, so that the sum of the cubes of these 8 terms will be reduced to $3^3 + 4^3 + 5^3$. In this case

$$\frac{p(p-1)(p+1)(p^2+2)}{6} = 6.$$

Hence, we have this remarkable result. $3^3 + 4^3 + 5^3 = 6^3$, which might have been obtained from the conditions $(18)^3 + (24)^3 + (30)^3 = (216)^3$ of the first problem, by dividing by 6^3.

When $p = 4$, we find, $6^3 + 7^3 + 8^3 + \dots + (68)^3 + (69)^3 = (180)^3$.

When $p = 5$, we have, $(34)^3\ (35)^3 + \dots + (157)^3 + (158)^3 = (540)^3$.

When $p = 10$, that is, when the number of terms $= p^3 = 1000$, we have $(1134)^3 + (1135)^3 + \dots + (2132)^3 + (2133)^3 = (16830)^3$.

REMARK.—In the *Mathematical Diary* for 1831, on page 186, it is stated that M. Pagliani had published his solution to this problem, in this last case, when the number of terms is 1000, in the "Annales de Mathematiques" by M. Gergonne.

In the *Mathematical Miscellany* for 1839, on page 127, William Lenhart has given a general solution of this problem, and as a particular case has obtained the same 1000 terms, as were given by M. Pagliani, and I wish, here, to express my high estimation of Mr. Lenhart's valuable contributions to this particular department of mathematics, given in the pages of the *Mathematical Miscellany*.

PROBLEM RELATING TO THE DETERMINATION OF CIRCULAR ORBITS.

BY G. W. HILL, ESQ., NYACK TURNPIKE, N. Y.

Determine the elements of the orbit of a planet or satellite, which moves in a circle in the plane of the ecliptic, from three observations of its direction from the earth, made at equal intervals of time; the positions of the earth and the central body at these times being known, but the sum of the masses of the central body and the planet or satellite being unknown.

Or, geometrically stated,—

In a plane, given a point as center and three straight lines, required to describe a circle, so that the arcs intercepted by the lines taken in a determinate order may be equal.

SOLUTION.

Let generally R denote the sun's distance from the earth,
" " L its longitude,
" " r the constant radius vector of the planet,
" " χ its heliocentric longitude,
" " η its heliocentric angular motion from one observation to the next,
" " λ its longitude as seen from the earth,
" " $\triangle$ its distance from the earth.

Moreover, employ the subscribts $(_{-1})$, $(_0)$, $(_1)$, to denote the special values of the above quantities, which have place at the three times of observation in their order.

By the theory of the transformation of rectangular co-ordinates from

the center of the sun as origin to the center of the earth, we shall have generally the two equations

$$\triangle \cos \lambda = r \cos \chi + R \cos L,$$
$$\triangle \sin \lambda = r \sin \chi + R \sin L.$$

From which may be derived the two

$$\triangle \cos (\lambda - P) = r \cos (\chi - P) + R \cos (L - P),$$
$$\triangle \sin (\lambda - P) = r \sin (\chi - P) + R \sin (L - P),$$

where P is any arbitrary angle. If we apply our equations to each of the three observations, we shall have the six equations

$$\begin{aligned}
\triangle_{-1} \cos \lambda_{-1} &= r \cos (\chi_0 - \eta) + R_{-1} \cos L_{-1},\\
\triangle_{-1} \sin \lambda_{-1} &= r \sin (\chi_0 - \eta) + R_{-1} \sin L_{-1},\\
\triangle_0 \cos \lambda_0 &= r \cos \chi_0 + R_0 \cos L_0,\\
\triangle_0 \sin \lambda_0 &= r \sin \chi_0 + R_0 \sin L_0,\\
\triangle_1 \cos \lambda_1 &= r \cos (\chi_0 + \eta) + R_1 \cos L_1,\\
\triangle_1 \sin \lambda_1 &= r \sin (\chi_0 + \eta) + R_1 \sin L_1.
\end{aligned}$$

These equations contain the six unknowns $\triangle_{-1}$, $\triangle_0$, $\triangle_1$, r, χ and η. If we eliminate $\triangle_{-1}$, $\triangle_0$, $\triangle_1$ from them, we shall have the three equations from which I started in my first solution.* But by retaining $\triangle_0$ as the unknown, we shall arrive at an elegant solution. Let us first then eliminate $\triangle_{-1}$ and $\triangle_1$; this we do by putting $P = \lambda_{-1}$ for the first two equations, and $P = \lambda_1$ for the last two. Our equations, for determining the four remaining unknowns, are

$$\begin{aligned}
0 &= r \sin (\chi_0 - \eta - \lambda_{-1}) + R_{-1} \sin (L_{-1} - \lambda_{-1}),\\
\triangle_0 \cos \lambda_0 &= r \cos \chi_0 + R_0 \cos L_0,\\
\triangle_0 \sin \lambda_0 &= r \sin \chi_0 + R_0 \sin L_0,\\
0 &= r \sin (\chi_0 + \eta - \lambda_1) + R_1 \sin (L_1 - \lambda_1).
\end{aligned}$$

If in the second and third of these equations we put successively $P = \eta + \lambda_{-1}$ and $P = -\eta + \lambda_1$, we get

$$\triangle_0 \sin (\lambda_0 - \eta - \lambda_{-1}) = r \sin (\chi_0 - \eta - \lambda_{-1}) + R_0 \sin (L_0 - \eta - \lambda_{-1}),$$
$$\triangle_0 \sin (\lambda_0 + \eta - \lambda_1 = r \sin (\chi_0 + \eta - \lambda_1) + R_0 \sin (L_0 + \eta - \lambda_1).$$

If from these equations we subtract the first and last of the preceding four, we get

$$\triangle_0 \sin (\lambda_0 - \eta - \lambda_{-1}) = R_0 \sin (L_0 - \eta - \lambda_{-1}) - R_{-1} \sin (L_{-1} - \lambda_{-1}),$$
$$\triangle_0 \sin (\lambda_0 + \eta - \lambda_1) = R_0 \sin (L_0 + \eta - \lambda_1) - R_1 \sin (L_1 - \lambda_1).$$

Behold us then, as the French say, arrived at two equations with two

*Mr. Hill here refers to a solution of this question communicated to Dr. Wright, and published in the *Yates County Chronicle* of February 5, 1874.—Ed.

unknowns, and that without complicating the form of our original equations.

It is very easy to eliminate Δ_0 from these, and we get

$$[R_0 \sin(L_0 - \eta - \lambda_{-1}) - R_{-1} \sin(L_{-1} - \lambda_{-1})] \sin(\lambda_0 + \eta - \lambda_1)$$
$$= [R_0 \sin(L_0 + \eta - \lambda_1) - R_1 \sin(L_1 - \lambda_1)] \sin(\lambda_0 - \eta - \lambda_{-1}).$$

But we prefer to keep Δ_0 as our final unknown. Let us put for the sake of brevity

$$\eta = \sigma + \frac{\lambda_1 - \lambda_{-1}}{2}, \quad \delta = \lambda_0 - \frac{\lambda_1 + \lambda_{-1}}{2}, \quad \delta^1 = L_0 - \frac{\lambda_1 + \lambda_{-1}}{2},$$

$$\psi_{-1} = L_{-1} - \lambda_{-1}, \quad \psi_1 = L_1 - \lambda_1.$$

All these are known quantities with the exception of σ, which will take the place of η as an unknown. Our two equations can now be written

$$\Delta_0 \sin(\delta - \sigma) = R_0 \sin(\delta^1 - \sigma) + R_{-1} \sin \psi_{-1},$$
$$\Delta_0 \sin(\delta + \sigma) = R_0 \sin(\delta^1 + \sigma) + R_1 \sin \psi_1.$$

Or by taking in succession the half the sum and half the difference

$$\Delta_0 \sin \delta \cos \sigma = R_0 \sin \delta^1 \cos \sigma + \frac{R_1 \sin \psi_1 + R_{-1} \sin \psi_{-1}}{2},$$

$$\Delta_0 \cos \delta \sin \sigma = R_0 \cos \delta^1 \sin \sigma + \frac{R_1 \sin \psi_1 - R_{-1} \sin \psi_{-1}}{2}.$$

$$\text{Whence } \cos \sigma = \tfrac{1}{2} \frac{R_1 \sin \psi_1 + R_{-1} \sin \psi_{-1}}{\Delta_0 \sin \delta - R_0 \sin \delta^1},$$

$$\sin \sigma = \tfrac{1}{2} \frac{R_1 \sin \psi_1 - R_{-1} \sin \psi_{-1}}{\Delta_0 \cos \delta - R_0 \cos \delta^1}.$$

By putting, (these are all known quantities)

$$a = \frac{R_1 \sin \psi_1 + R_{-1} \sin \psi_{-1}}{2 \sin \delta}, \; b = \frac{R_1 \sin \psi_1 - R_{-1} \sin \psi_{-1}}{2 \cos \delta},$$

$$c = R_0 \frac{\sin \delta^1}{\sin \delta}, \; d = R_0 \frac{\cos \delta^1}{\cos \delta},$$

we shall obtain the very elegant form for our final equation determining Δ_0,

$$\left\{\frac{a}{\Delta_0 - c}\right\}^2 + \left\{\frac{b}{\Delta_0 - d}\right\}^2 = 1.$$

This is, as we see, of the fourth degree in Δ_0; in the form in which

Problem 400† is stated, this equation will be found to have a root $\Delta_6 = 0$, that is the absolute term of the equation will be 0; in this case therefore the equation reduces to the third degree.

By the introduction of the new unknown

$$x = \Delta_6 - \frac{c+d}{2}, \text{ and putting } h = \frac{c-d}{2}$$

the equation takes the somewhat simpler form

$$\left\{\frac{a}{x+h}\right\}^2 + \left\{\frac{b}{x-h}\right\}^2 = 1,$$

$$\text{or } (x^2 - h^2)^2 = a^2(x-h)^2 + b^2(x+h)^2.$$

SOLUTIONS OF PROBLEMS IN NO. 1.

In this department we will in general publish but one solution to each problem proposed, though, in some cases, when the method pursued in the solutions is essentially different, two or more solutions of the same question will be published. Credit will be given, however, in each number to all who shall have furnished correct solutions of the questions whose solutions are published in that number.

Each solution should contain sufficient detail to be comprehended by the ordinary reader who is acquainted with the elements of the branches employed in the solution. Clearness must not be sacrificed to brevity, but, other things being equal, the brevity of a solution will determine its selection for publication.

When several persons furnish essentially the same solution to a question, his name only will be placed at the head of the published solution whose notation and phraseology are adopted.

Persons sending solutions are requested to put the solution of *each question*, together with the name of the writer, on a separate piece of paper.

Solutions have been received as follows:

R. M. DeFrance solved 1 and 2; Theo. L. DeLand solved 1; Prof. A. B. Evans solved 1, 2, 3 and 4; Prof. C. Hornung solved 2; Philip Hoagland solved 2; Henry Heaton solved 1, 2, 3 and 4; Prof. E. W. Hyde solved 3; Prof. Knisely solved 1; Miss Esther W. Matthews, (State Normal School, Kirksville, Mo.), solved 1; Artemas Martin solved 1, 2, 3 and

†*Yates County Chronicle.*

4; O. D. Oathout solved 2; L. Regan solved 1 and 2; E. B. Seitz solved 1, 2, 3 and 4; Elias Schneider, A. M., solved 1 and 2; S. W. Salmon solved 1, 2, 3 and 4; Prof. D. W. Sensenig solved 1, 2 and 4; Walter Siverly solved 1, 2, 3 and 4, and John M. Wilt, A. M., solved 1 and 4.

---

1. "Find the values of x and y in the following equations:

$$a^2 x^4 + b^2 y^4 = a^2 b^2 (x + y)^2; \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$
$$a^2 x^2 + b^2 y^2 = a^2 b^2." \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

SOLUTION BY THEO. L. DE LAND, WASHINGTON, D. C.

Divide (1) by (2), expand $(x+y)^2$, clear of fractions, cancel like terms, factor, divide by $x\,y$, substitute $a^2\,b^2$ for its value, multiply by $2\,a\,b$, and we have

$$2\,a\,b\,x\,y = -\frac{4\,a^3\,b^3}{a^2+b^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Both add to, and subtract from (2), eq. (3), and we have

$$(a\,x \pm b\,y)^2 = a^2\,b^2 \mp \frac{4\,a^3\,b^3}{a^2+b^2}.$$

The upper signs give one equation and the lower signs another.

$$\text{Hence, } x = \pm \frac{b}{2\sqrt{a^2+b^2}}\left(\sqrt{(a-b)^2-2\,a\,b} \pm \sqrt{(a+b)^2+2\,a\,b}\right)$$

$$\text{and } y = \pm \frac{a}{2\sqrt{a^2+b^2}}\left(\sqrt{(a-b)^2-2\,a\,b} \mp \sqrt{(a+b)^2+2\,a\,b}\right).$$

[Prof. Knisely writes that the question he sent (No. 1) is, by mistake slightly different from what he intended to present, which, he says, "arises from the problem of a tangent to an ellipse, the tangent being intercepted between the axes produced, and equal to the sum of the semi-axes." The question, as he intended to present it, and his solution are subjoined:]

$$"a^2\,x^2 + b^2\,y^2 = x^2\,y^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$
$$b^2\,y^4 + a^2\,x^4 = a^2\,b^2\,(x + y)^2. \quad . \quad . \quad . \quad . \quad (2)$$

SOLUTION.

$$\text{From (1) } y^2 = \frac{a^2\,x^2}{x^2 - b^2}, \quad y = \frac{a\,x}{\sqrt{x^2 - b^2}}.$$

Hence, in (2),

$$a^2\,x^4 + \frac{a^4\,b^2\,x^4}{(x^2-b^2)^2} = a^2\,b^2\left(x^2 + \frac{2\,a\,x^2}{\sqrt{x^2 - b^2}} + \frac{a^2\,x^2}{x^2 - b^2}\right).$$

Dividing by $a^2\,x^2$,

$$x^2 + \frac{a^2\,b^2\,x^2}{(x^2-b^2)^2} = b^2 + \frac{2\,a\,b^2}{\sqrt{x^2 - b^2}} + \frac{a^2\,b^2}{x^2 - b^2}.$$

$$x^2\,(x^2-b^2)^2 + a^2\,b^2\,x^2 = (x^2-b^2)^2\,b^2 + 2\,a\,b^2\,(\sqrt{x^2-b^2})^3 + (x^2-b^2)\,a^2\,b^2..$$

$$x^6 - 2x^4b^2 + b^4x^2 + a^2b^2x^2 = b^2x^4 - 2b^4x^2 + b^6 + 2ab^2(\sqrt{x^2-b^2})^3$$
$$+ a^2b^2x^2 - a^2b^4$$

$$x^6 - 3x^4b^2 + 3x^2b^4 - b^6 - 2ab^2(\sqrt{x^2-b^2})^3 + a^2b^4 = 0.$$

$$(x^2-b^2)^3 - 2ab^2(x^2-b^2)^{\frac{3}{2}} + a^2b^4 = 0.$$

Taking square root,

$$\sqrt{(x^2-b^2)^3} = ab^2. \quad x^2 = (ab^2)^{\frac{2}{3}} + b^2. \quad y^2 = (a^2b)^{\frac{2}{3}} + a^2.$$

—U. Jesse Knisely."

---

2. "Let a regular polygon of 14 sides be described, each of whose equal sides shall be *one*. Then will the radius of its circumscribing circle, which put$=r$, be more than *two* and less than *three*. Put $r=2+x$; then is x a positive quantity less than *one*. Let another regular polygon of half the number of sides (7) be inscribed in a circle whose radius is *one*, and determine one of its equal sides in functions of x expressed in its simplest form."

SOLUTION BY PROF. ASHER B. EVANS, LOCKPORT, N. Y.

Let AB represent a side of the regular polygon of 14 sides, and O the center of its circumscribing circle. Then

$$AB = 1 : AO = r :: \sin \tfrac{1}{7}\pi : \sin \tfrac{3}{7}\pi;$$

$$\therefore AO = r = \frac{\sin \frac{3}{7}\pi}{\sin \frac{1}{7}\pi} = 3 - 4\sin^2 \tfrac{1}{7}\pi \quad . \quad . \quad . \quad (1)$$

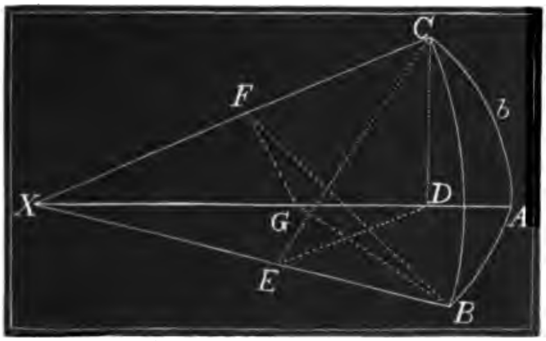
Since $2\sin \frac{1}{7}\pi$ is less than $2\sin 30°$, that is, less than unity, $4\sin^2 \frac{1}{7}\pi$ is positive and less than unity, and equation (1) gives $r > 2$ and < 3.

If $r=2+x$, $x=1-4\sin^2 \frac{1}{7}\pi$, or $2\sin \frac{1}{7}\pi = \sqrt{1-x}$. But $2\sin \frac{1}{7}\pi$ is the expression for a side of a regular polygon of 7 sides the radius of the circumscribing circle being unity; therefore the required function of x is $\sqrt{1-x}$.

---

3. "If a line make an angle of 40° with a fixed plane, and a plane embracing this line be perpendicular to the fixed plane, how many degrees from its first position must the plane embracing the line revolve in order that it may make an angle of 45° with the fixed plane?"

SOLUTION BY PROF. E. W. HYDE, CHESTER, PA.



Let b = angle between line and plane; C = angle through which plane through line must revolve, and B = angle plane makes with fixed plane; then we have at once, by Napier's formulæ,

$$\cos B = \cos b \sin C;$$

$$\therefore \sin C = \frac{\cos B}{\cos b},$$

and if $b = 40°$ and $B = 45°$,

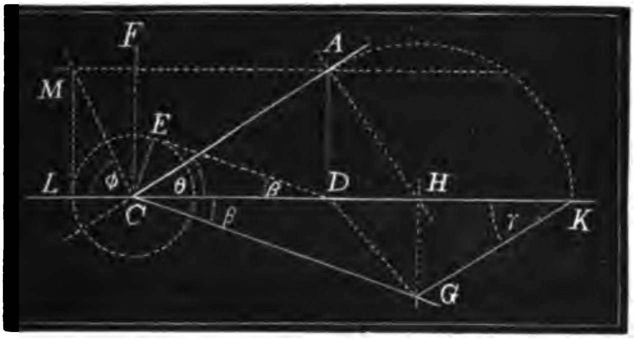
$$\sin C = \frac{\cos 45^\circ}{\cos 40^\circ} = \frac{1}{\sqrt{2}\cos 40^\circ};$$

$$\therefore C = 67^\circ\ 22'\ 40''.$$

[All the solutions of No. 8 which have been received, except Mr. Salmon's, are analogous to the above.]

SOLUTION BY S. W. SALMON, MOUNT OLIVE, N. J.

Take the fixed plane as the horizontal plane of projection (H). Let the given line CA make an angle θ with H, and take the vertical plane through this line as the vertical plane of projection (V). Let C be the point in which CA pierces H. Let γ be the angle through which the vertical plane through CA has to revolve in order to make an angle ϕ with H, and let β be the angle which the horizontal trace of this plan makes with the ground line.



Draw CF perpendicular to H, and let it be the axis of a cone with a circular right section whose vertex is C and whose elements make an angle ϕ with H. Through CA pass a plane tangent to this cone; the tangent plane will then make an angle ϕ with H; CA is the vertical trace of this plane. In order to find the horizontal trace pass a plane parallel to H through A; it cuts a circle from the cone and a line from the plane tangent to the circle. ED is the horizontal projection of this line, and CG, drawn parallel to ED, is the horizontal trace of the plane. Through A pass a plane perpendicular to CA, AH is its vertical and GH its horizontal trace; it cuts a line from the plane GCA, the position of which when revolved around GH into the horizontal plane is GK. The angle $GKH = \gamma$.

$$LM = AD = CD \operatorname{tang} \theta$$

$$\text{and } LC = CE = \frac{LM}{\operatorname{tang}\phi} = \frac{CD \operatorname{tang}\theta}{\operatorname{tang}\phi} = CD \sin\beta,$$

$$\text{whence } \beta = \sin^{-1}\frac{\operatorname{tang}\theta}{\operatorname{tang}\phi};$$

$$\operatorname{tang}\gamma = \frac{GH}{HK} = \frac{CH \operatorname{tang}\beta}{CH \sin\theta} = \operatorname{tang}\frac{\left(\sin^{-1}\dfrac{\operatorname{tang}\theta}{\operatorname{tang}\phi}\right)}{\sin\theta}.$$

$$\therefore \gamma = \text{tang}^{-1}\left[\frac{\text{tang}\left(\sin^{-1}\frac{\text{tang}\,\theta}{\text{tang}\,\phi}\right)}{\sin\theta}\right].$$

When $\theta = 40°$ and $\phi = 45$, $\gamma = 67°\ 22'\ 41''$.

[This question admits of still another solution, as follows: In the figure to Prof. Hyde's solution, above, draw CD perpendicular to AX and DE perpendicular to BX. Then, because the $< CED = 45°$, $DE = CD$. $\therefore$ (if $CX = 1$) $CE = \sqrt{2}\sin 40°$, and $EX = \sqrt{1-2\sin^2 40°}$. Make $FX = EX$; then is $FG = \tan 40° \sqrt{1-2\sin^2 40°}$. Join BG and BF. Then is BGF a right angbled triangle, right angle at G, and $BF = CE$. $\therefore$ we have $BF : FG ::$ radius : cos of the requirad angle, or, $\sqrt{2}\sin 40° : \tan 40° \sqrt{1-2\sin^2 40°} :: 1 : \cos GFB = 67°\ 22'\ 41''$, very nearly.]

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4. "A cask containing a gallons of wine stands on another containing a gallons of water; they are connected by a pipe, through which, when open, the wine can escape into the lower cask at the rate of c gallons per minute, and through a pipe in the lower cask the mixture can escape at the same rate; also, water can be let in through a pipe on the top of the upper cask at a like rate. If all the pipes be opened at the same instant, how much *wine* will be in the lower cask at the end of t minutes, supposing the fluids to mingle perfectly?"

SOLUTION BY S. W. SALMON, MOUNT OLIVE, N. J.

Let Q be the quantity of wine in the lower cask at the end of t minutes; and let Q_1, Q_2, Q_3 .... Q_n be the quantity at the end of the 1st 2d, 3d .... nth instants, q, the quantity of the mixture that escapes in an instant, and n, the number of instants in a minute, then is $nq = c$,

$$Q_1 = q = q\left(1-\frac{q}{a}\right)^0,\quad Q_2 = 2q\left(1-\frac{q}{a}\right),$$

$$Q_3 = 3q\left(1-\frac{q}{a}\right)^2 \ldots\ldots Q_n = nq\left(1-\frac{q}{a}\right)^{n-1}.$$

$$\therefore Q = tnq\left(1-\frac{q}{a}\right)^{tn-1}.$$

Taking the Napierian logarithm of both members of this equation,

$$\log Q = tn\log\left(1-\frac{q}{a}\right) + \log ct$$

$$= \log c\,t + t\,n\left(-\frac{q}{a} - \frac{q^2}{2\,a^2} - \frac{q^3}{3\,a^3} - \text{\&c.}\right).$$

Neglecting powers of infinitesimals, we have

$$\log\left(\frac{c\,t}{Q}\right) = \frac{t\,n\,q}{a} = \frac{c\,t}{a}.$$

$$\therefore e^{\frac{ct}{a}} = \frac{c\,t}{Q}. \quad \therefore Q = c\,t\,e^{-\frac{ct}{a}};$$

where e denotes the base of the Napierian system of logarithms.

[This question was solved by Prof. Evans in an elegant manner by the method of finite differences, and in nearly the same manner as the above by Prof. Sensenig. All the other solutions were by application of the differential and integral calculus.]

QUERY.—Can a demonstration be given of the following formula for primes?

$$N = \frac{x}{A \log x - B};$$

in which N denotes the number of prime numbers contained in any number x, and A and B are constants.—Communicated by PROF. EDWARD BROOKS.

NOTE ON SUN SPOTS.—One might easily gather from reading astronomical works, that solar spots are rarely visible to the naked eye; that is, without the use of a telescope; but such is not the case. Let any one who feels an interest in the subject, prepare a suitable smoked glass, and examine the sun's disk daily, or as often as the clouds will permit, and he will find that solar spots can frequently be seen without a telescope. If it were worth the space to record the observations, I could give numerous instances when I saw spots without a telescope, and I have occasionally seen two at a time.

Since the sun's spots return periodically once in about eleven years; that is, from the minimum average number they gradually increase in number and area till the maximum is reached in about five and a half years, when the number gradually decreases; large spots are more likely to be seen about the time of the maximum number.

To discover a solar spot without a telescope, keep the eye directed atten-

tively on the sun (always using a smoked glass, or something equivalent) for a minute, and sometimes for several minutes, and the probability is that a spot, which at first could not be seen, will become visible.—Communicated by PROF. D. TROWBRIDGE.

QUERY.—The strain on the internal surface of a steam boiler tending to produce longitudinal rupture, is, per unit of length, equal to the product of the elastic pressure of the steam per unit of surface multiplied by the diameter of the boiler. What should be the relative thickness of the flat ends of a boiler, so that, without bracing, they are just strong enough to withstand the maximum strain that the boiler will bear?

PROBLEMS.

11. Borrowed a sum of money at 8 per cent. simple interest and loaned it out again at 5 per cent. compound interest; in what time will I gain the amount borrowed?—Communicated by E. W. FOSDICK, ESQ., Butler, Indiana.

12. Given the base $A\ C$ of a triangle and the ratio of $A\ B$ to $B\ C$ to find the locus of the point B by Geometry.—Communicated by L. REGAN, Boonsboro, Iowa.

13. A body is projected at a given distance from the center of force with a given velocity, and in a direction perpendicular to that distance: When the force is repulsive and varies inversely as the cube of the distance, find the path of the body:—Communicated by J. B. SANDERS, Troy, Indiana.

14. Two equal particles attracting each other with forces varying inversely as the square of the distance, are constrained to move in two straight lines at right angles to each other; supposing their motions to commence from rest, to find the time in which each of them will arrive at the intersection of the two straight lines.—Communicated by PROF. J. M. GREENWOOD, Kirksville, Mo.

15. Two points are taken at random in the surface of a given circle, and a chord drawn through each at random; show that the chance that the chords intersect is

$$\frac{1}{3} + \frac{5}{2\pi}.$$

—Communicated by E. B. SEITZ, Greenville, Ohio.

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REMARKS ON THE STABILITY OF PLANETARY SYSTEMS.

BY G. W. HILL, ESQ., NYACK TURNPIKE, N. Y.

As, in some quarters, quite erroneous views seem to be entertained regarding the conditions necessary for the stability of the solar system, it may be of service to note here, in brief, what is known on this subject.

It is remarkable that, although the meaning of stability in statics is well known, no one, so far as I know, has ever given a rigorous definition of this term as used in dynamics. As applied to the solar system, the sense attributed to it in general seems to involve the idea that the mean distances, eccentricities and mutual inclinations of the planets should always be comprised within narrow limits. But if this be the proper meaning of the word, one is tempted to ask—how narrow? It is plain, when we consider the matter more closely, that the distinction between stability and instability is one of kind and not of degree. There must be a sharp line separating stable systems from unstable.

In the first place we must discriminate between two possible significations of the term; a system may be stable or unstable with reference to the action of foreign forces, or with reference to the mutual action of its parts. A slight disturbance from without may cause in a moving system only trifling deviations from the previous paths of motion, or the effect may be a greater and still greater departure from them. This is quite analogous to the stability and instability of statics. But the stability of a planetary system, with reference to its own action, must be defined in a way quite peculiar.

A planetary system is stable when finite superior and inferior limits can be assigned to all the distances of the bodies composing it, and that, no matter how long the motion may be prolonged; but if to some or all the distances, no superior limit other than infinity, or no inferior limit other than zero, can be assigned, the system is unstable.

$$\frac{2\,V\lambda}{T''},\ \frac{2\,V'}{t''},\ \frac{2\,V'}{\tau''} =$$

equatorial g at Sun, Earth and Jupiter.

$\frac{1}{32}\pi^{2\nu}$ = ratio of the integral of infinitesimal impulses during revolution in a circular orbit π^{ν}, to the integral of similar impulses during fall from circumference to center of same orbit.

$\frac{1}{32}\pi^{6}$ = Neptune's mean distance from Sun, in units of Earth's mean distance.

$\frac{1}{32}\pi^{5}$ = Saturn's mean distance.

$\frac{1}{32}\pi^{4}$ = Asteroidal mean distance, or twice mean distance of Mars.

$\frac{1}{32}\pi^{3}$ = Earth's secular mean perihelion distance.

$\frac{1}{32}\pi^{2}$ = Mercury's secular mean perihelion distance.

$\frac{1}{32}\pi^{-1}$ = major axis of Sun's orbit about center of gravity of binary star.

$\frac{1}{32}\pi^{-2}$ = heliocentric distance of linear center of oscillation of secular mean perihelion center of gravity of the binary star.

The ratio of V' to V'' is found by supposing Sun's radius to vary from r to $n^2 r$. Under such variations

$$V' \propto \frac{1}{n}; \quad V'' \propto \frac{1}{n^2}.$$

In the following table A represents the theoretical values as estimated from $V\lambda$; B, for Jupiter's distance, $\varDelta$,

$$V'' = \frac{2\,\pi\,\tau'}{\varDelta};$$

C the observed values. For T'', C is the mean of the several estimates published by Bianchi and Laugier, Lelambre, Petersen, Sporer, Carrington and Faye. The Sun's annual motion is given in units of Earth's radius vector, C being Struve's estimate. For V', A, B and C are deduced from g on Sun, Earth and Jupiter:

	A	B	C
T''	22,036.45 sec.	21,639.07 sec.	21,628.02 sec.
V'	265.66 m.	261.39 m.	261.56 m.
$4\ V'' \times 1$ yr.	1.678 r. v.	1.754 r. v.	1.623 r. v.

The $\frac{1}{32}\pi^{2\nu}$ series groups the principal planets into four pairs. The correspondence between the theoretical and observed values is given below in units of Sun's radius. The values of the secular mean apsides are taken from "Stockwell's Memoirs on the Secular Variations of the Orbits of the Eight Principal Planets."

	Theoretical.	Observed.
Neptune, mean,	6,450.776	6,453.751
Saturn, mean,	2,053.346	2,049.514
Asteroid, mean,	653.600	654.760
Earth, perihelion,	208.048	207.583
Mercury, perihelion,	66.224	68.483
Sun, major axis,	2.136	2.132
Primary center, osc.,	.679	.679

The slight discrepancies in the values of T'', V', V'', seem to be attributable to Jupiter's mean eccentricity, but they are all within the limits of uncertainty of observation.

PERFECT CUBES.

BY PROF. W. D. HENKLE, SALEM, OHIO.

The following interesting facts in reference to perfect cubes were discovered by the writer about seven years ago:

Perfect cubes ending in 1, 2, 3, 4, 5, 6, 7, 8, 9 or 0 followed by 1
have roots ending in 7, 4, 1, 8, 5, 2, 9, 6, 3 or 0 followed by 1.

Perfect cubes ending in 1, 2, 3, 4, 5, 6, 7, 8, 9 or 0 followed by 3
have roots ending in 1, 4, 7, 0, 3, 6, 9, 2, 5 or 8 followed by 7.

Perfect cubes ending in 1, 2, 3, 4, 5, 6, 7, 8, 9 or 0 followed by 7
have roots ending in 7, 0, 3, 6, 9, 2, 5, 8, 1 or 4 followed by 3.

Perfect cubes ending in 1, 2, 3, 4, 5, 6, 7, 8, 9 or 0 followed by 9
have roots ending in 3, 0, 7, 4, 1, 8, 5, 2, 9 or 6 followed by 9.

Hence every perfect cube ending in 11 has 71 as the ending of its root; every perfect cube ending in 21 has 41 as the ending of its root; and so on. Thus we see that the last two figures of any perfect cube ending in 1, 3, 7 or 9 may be known if the figure before 1, 3, 7 or 9 is also known. For instance the cube root of the perfect cube 185193 may be known to be 57 because all perfect cubes ending in 93 have roots ending in 57. Thus we may know the cube root of the perfect cube 3,869,893 by knowing only 3, ... 93. The first figure of the root must be 1 and the next two 57.

If the series 1, 2, 3, 4, 5, 6, 7, 8, 9, 0 be repeated or made circular it

may be seen that the series 7, 4, 1, 8, 5, 2, 9, 6, 3, 0 may be obtained by beginning at 7 in the circular series and counting forward successively to the 7th following figure; in the same way the series 1, 4, 7, 0, 3, 6, 9, 2, 5, 8 can be found by counting forward by threes beginning at 1. To get the series 7, 0, 3, 6, 9, 2, 5, 8, 1, 4 begin with 7 and count by threes. To get the series 3, 0, 7, 4, 1, 8, 5, 2, 9, 6 begin at 3 and count by sevens.

It is desirable to obtain some method of getting directly the ten's figure of the cube. The following scheme accomplishes this:

1	2	3	4	5	6	7	8	9	0
7	7	7	7	7	7	7	7	7	7
7	.4	.1	.8	.5	.2	.9	.6	.3	0

1	2	3	4	5	6	7	8	9	0
3	3	3	3	3	3	3	3	3	3
3	6	9	.2	.5	.8	.1	.4	.7	0
8	8	8	8	8	8	8	8	8	8
.1	.4	.7	.0	.3	.6	.9	.2	.5	8

1	2	3	4	5	6	7	8	9	0
3	3	3	3	3	3	3	3	3	3
3	6	9	.2	.5	.8	.1	.4	.7	0
4	4	4	4	4	4	4	4	4	4
7	.0	.3	.6	.9	.2	.5	.8	.1	4

1	2	3	4	5	6	7	8	9	0
7	7	7	7	7	7	7	7	7	7
7	.4	.1	.8	.5	.2	.9	.6	.3	0
6	6	6	6	6	6	6	6	6	6
.3	.0	.7	.4	.1	.8	.5	.2	.9	6

In the first case multiply by 7 and reject the left-hand digit when there is one; in the second case multiply 3 and add 8; in the third multiply 3 and add 4; and in the fourth multiply 7 and add 6, and reject as before when necessary.

The process may be stated in another way: Putting D as the ten's figure of the cube, and sub-figures as the last figures of the cubes.

To find the ten's figure of the root of a cube ending in D_1 multiply D by 7 and cast out the tens.

When the cube ends in D_3, multiply D by 3, add 8, and cast out the tens.
" " " " D_7, " " 3, add 4, " " "
" " " " D_9, " " 7, add 6, " " "

For instance to find the cube root of 912673: The last figure of the root must be 7; to get the ten's figure multiply 7 by 3, add 8, and cast out the tens or reject the left-hand digit; this gives 9, hence the root is 97.

$$\text{Perfect cubes ending in } 1,\ 3,\ 5,\ 7, \text{ or } 9 \text{ followed by } 2$$
$$\text{have roots ending in} \left\{ \begin{matrix} 5, & 1, & 7, & 3, & \text{or } 9 \\ 0, & 6, & 2, & 8, & \text{or } 4 \end{matrix} \right\} \text{followed by } 8.$$

$$\text{Perfect cubes ending in } 2,\ 4,\ 6,\ 8, \text{ or } 0 \text{ followed by } 4$$
$$\text{have roots ending in} \left\{ \begin{matrix} 7, & 1, & 5, & 9, & \text{or } 3 \\ 2, & 6, & 0, & 4, & \text{or } 8 \end{matrix} \right\} \text{followed by } 4.$$

$$\text{Perfect cubes ending in } 1,\ 3,\ 5,\ 7, \text{ or } 9 \text{ followed by } 6$$
$$\text{have roots ending in} \left\{ \begin{matrix} 5, & 9, & 3, & 7, & \text{or } 1 \\ 0, & 4, & 8, & 2, & \text{or } 6 \end{matrix} \right\} \text{followed by } 6.$$

$$\text{Perfect cubes ending in } 2,\ 4,\ 6,\ 8, \text{ or } 0 \text{ followed by } 8$$
$$\text{have roots ending in} \left\{ \begin{matrix} 6, & 2, & 8, & 4, & \text{or } 0 \\ 1, & 7, & 3, & 9, & \text{or } 5 \end{matrix} \right\} \text{followed by } 2.$$

In these cases there are alternative series. In the case of D_2 the corresponding number of the first of the alternative series can be obtained by multiplying D by 3, and adding 2, and rejecting the tens; in the case of D_4 and D_6, by multiplying D by 2, adding 3, and rejecting the tens, and in the case of D_8, by multiplying D by 3, and rejecting the tens.

To obtain the corresponding alternative of the second series subtract 5 when possible, if not add 5.

What is the cube root of the perfect cube 110592? Multiplying 9 by 3, adding 2, and rejecting the tens we have 9. The alternative is 5 less, or 4. Hence the root is either 98 or 48. The 110 shows that it must be 48. What is the cube root of the perfect cube 25,412,184? Multiplying 8 by 2, adding 3, and rejecting the tens we get 9. Hence the ten's figure of the root is either 9 or 4. From 25 we get the first figure 2, and from 4 the last figure 4. The root is either 294 or 254. We decide in favor of 294 because 25 is nearly the cube of 3. There is another mode of deciding which is the root: Assume either one as the cube root, subtract it from the cube and the remainder is divisible by 6 if the root assumed is right. A cube when divided by 6 gives the same remainder as when its root is divided by 6.

$$\text{Perfect cubes ending in } 2 \text{ or } 7 \text{ followed by } 5$$
$$\text{have roots ending in} \left\{ \begin{matrix} 0 \text{ or } 1 \\ 2 \text{ or } 3 \\ 4 \text{ or } 5 \\ 6 \text{ or } 7 \\ 8 \text{ or } 9 \end{matrix} \right\} \text{followed by } 5.$$

This is the most difficult case. When a perfect cube ends in 25 its root must end in an even number(0 being included) followed by 5. If a perfect cube ends in 75, its root must end in an odd number followed by 5. Find the cube root of 61,629,875. The root is 315, 335, 355, 375 or 395. Dividing 61,629,875 by 6 we get 5 for a remainder. The same remainder is obtained from 335 and 395. Hence we have only to decide between 335 and 395. The 61 plainly indicates 395.

In a future article I shall discuss the relation of the hundreds of the cube to the hundreds of the root.

LETTER FROM PROF. SCHIAPARELLI.

EDITOR OF THE ANALYST, DES MOINES, IOWA:—Prof. A Hall of Washington, has had the kindness to send me No. 2 of your journal, THE ANALYST, containing an article with the title "Comets and Meteors." In this article Mr. Hall offers some objections to the difficulties which I have opposed to the calculation of Laplace on the probabilities of hyperbolic orbits, (*Conn. des Temps* for 1816, pp. 215–218). I ask of you permission to add some observations in reply to the remarks of Mr. Hall.

The first remark is thus stated: "If S be the sun and P the point on the surface of the sphere of activity of the sun's attractive force, the bodies that pass through P may have all possible directions, but a direction making a small angle with PS is less probable than one making a greater angle. I cannot see that Prof. Schiaparelli has introduced this condition into his solution." I have always been persuaded of the truth of the proposition announced by Mr. Hall. If I have not explicitely spoken of it, it is only because the course of the demonstration does not require it. My solution would have been erroneous if I had supposed the contrary proposition; that is to say, that all the angles with $P\,S$ are equally probable. Mr. Hall will have some difficulty to prove that I have committed an error of this gravity. In order to establish that a demonstration is in fault it is not sufficient to say that this or that idea is not indicated; it is necessary to find some evident error either in the fundamental supposition or in the logical connection of the reasoning.

The second remark of Mr. Hall refers to a detail of calculation; that is to say, to this, that in seeking to find the defect of the solution of Laplace I have not spoken of the division by U, indicated by this great geometer at the beginning of page 216 of his memoir. I shall observe here that U is the greatest value physically attributable to the velocity v; it is there

fore a constant limit. As it is required here not to calculate the *absolute* probabilities but only their *ratios*, it is quite indifferent whether we introduce or not the common divisor U. This is so true that when Laplace compares the two probabilities he finds for their ratio the equation, (page 218),

$$\frac{\pi-2}{100}\sqrt{\frac{r}{2D}}\,(r+200)-1:1,$$

from which U has completely disappeared. As in my investigation I sought the essential point, I was not occupied with this division by U, to which I have not the least objection since it is wholly foreign to our question.

My difficulties remain therefore intact; they are represented in the two following propositions:

I. The integral $\int_a^\infty d\,v\left[1-\frac{\sqrt{1-\frac{D}{r}}}{r\,v}\sqrt{r^2\,v^2\left(1+\frac{D}{r}\right)-2\,D}\right]$ ..(a)

is infinite whatever may be the finite value of the lower limit a.

II. By means of an incomplete development into a series Laplace has found for this quantity a finite value, which has completely changed the final result of his research.

So long as one has not demonstrated the inexactness of the one or the other of these two propositions I cannot regard as destroyed the objections that I have raised against the analysis of Laplace. I will add that it is erroneous in another place. In fact, in evaluating the probabilities Laplace excludes all the bodies for which the values of v are comprised between the limits

$$v=0,\text{ and }v=\frac{\sqrt{2D}}{r\sqrt{1+\frac{D}{r}}},\quad\ldots\ldots\ldots\ldots\ldots\ldots(b)$$

apparently for the reason that he derives an imaginary value for the second of the radicals contained in the expression (a). But this is not a sufficient reason. In fact, when v is comprised between the limits indicated, the imaginary value of (a) signifies that the perihelion distance of the orbits described by these bodies cannot be equal to D, and that it is always less than D. These bodies ought therefore to enter into the account in determining the probability; only for them it is necessary to calculate the probability in another manner.

I must add, in order to be just, a remark which escaped me in 1871, at the time of the publication of my book on the meteoric stars. I say that

these two objections that I have found to the investigation of Laplace, although theoretically established, are not of a great practical consequence for the resolution of his problem. In fact, as to that which regards the omission of the velocities comprised between the limits (b), it is easy to show that these velocities always correspond to elliptic orbits. It follows that in rectifying this omission we shall augment the number of orbits very elongated, which, according to Laplace, ought to be the most numerous; which would only reinforce his conclusion.

As to the error in the evaluation of the integral (a), it exists only when we extend the upper limit to infinity; that is to say, when we suppose equally probable all the velocities from 0 even to ∞. This supposition is physically inadmissable; it is necessary therefore to give a finite value to the upper limit of the integration. In putting for this limit a moderate value of v, as 10, 20 or 50, we find that the integral (a) does not differ much from the value assigned by Laplace when extending it even to infinity, one error corrects the other in such a way that practically his final conclusion appears to be very nearly justified.

In resuming, the inadvertencies which I have noticed only affect the mathematical perfection of the problem; but they do not really change much the practical value of the final conclusion drawn by Laplace. This is what I should have recognized sooner, viz: That the true cause of the error of this final conclusion must be sought for in the fundamental suppositions, and precisely in this fact that Laplace has neglected to consider the motion of the solar system in space. In 1813 the astronomers had not much confidence in the speculations of W. Herschel on the proper motion of the solar system; they could therefore reasonably exclude it from consideration. This is not permitted to-day. In reconsidering the problem therefore under the point of view of Laplace, but with the supposition that the solar system is transported in space with the veloclty u, we shall find not only a very great excess of probability in favor of orbits strongly hyperbolic; but we shall see moreover that the hyperbolas whose axes approach this quantity

$$-\frac{1}{u^2}$$

must be more frequent than others. This being contrary to observation, we must conclude that the comets are not bodies of a stellar nature.

Accept, Sir, the expression of my sincere respect,

J. V. SCHIAPARELLI.

Director of the Observatory, Milan, Italy.

THE BESSELIAN FUNCTION,

BY PROF. ASAPH HALL.

The investigation in the theory of elliptic motion of the relations between the different anomalies, the expression of the radius-vector by means of series, and the expansion into series of various functions of the radius-vector and the anomalies, have led to interesting and valuable results. The theorem of Lagrange has here a direct application, but the method of determining the coefficients by means of definite integrals is more elegant. This method appears to have been first indicated by Poisson in the first edition of his Mecanique, but it was first completely worked out by Bessel in a memoir published among the memoirs of the Berlin Academy for 1816–17.

Let ε and g be the excentric and the mean anomalies, and e the excentricity of the ellipse, then

$$g = \varepsilon - e \sin \varepsilon.$$

Assume $\varepsilon - g = A_1 \sin g + A_2 \sin 2\,g + \ldots. + A_i \sin i\,g + \ldots.$ (1)

If i and i' are two whole numbers, we shall have when they are unequal

$$\int_0^\pi \cos i\,g \cos i'\,g.\,d\,g = 0, \quad \int_0^\pi \sin i\,g \sin i'\,g\,.\,d\,g = 0,$$

and when $i = i'$,

$$\int_0^\pi \cos^2 i\,g\,.\,d\,g = \tfrac{1}{2}\,\pi, \quad \int_0^\pi \sin^2 i\,.\,d\,g = \tfrac{1}{2}\,\pi.$$

Multiplying equatien (1) by $\sin i\,g\,.\,d\,g$ and integrating, we shall have

$$A_i = \frac{2}{\pi}\int_0^\pi (\varepsilon - g) \sin i\,g\,.\,d\,g.$$

It is more convenient to use the variable ε, and integrating by parts' since $\varepsilon - g$ is zero at the limits o and π, we have

$$A_i = \frac{2}{i\pi}\int_0^\pi \cos i\,g\left(\frac{d\,\varepsilon}{dg} - 1\right).\,d\,g,$$

$$\text{or since} \int_0^\pi \cos i\,g\,.\,d\,g = 0,$$

$$A_i = \frac{2}{i\pi}\int_0^\pi \cos i\,g\,.\,d\,\varepsilon = \frac{2}{i\pi}\int_0^\pi \cos i\,(\varepsilon - e \sin \varepsilon)\,.\,d\,\varepsilon \ldots..(2)$$

For the radius-vector we have a series proceeding according to the cosines of multiples of the mean anomaly, and the coefficients in this series can be derived from the value of A_i by a simple differentiation

with respect to the excentricity. In the series for the equation of center the coefficients can also be derived from the value of A_i, but by a more complicated relation.

In order to show another occurrence of this definite integral let it be required to change a series proceeding by sines and cosines of multiples of ε into one proceeding according to sines and cosines of multiples of g, a problem that occurs in the theory of perturbations. Here he have to solve the equations

$$\sin m\,\varepsilon = \Sigma\, A_i^m \sin i\,g; \qquad \cos m\,\varepsilon = \Sigma\, B_i^m \cos i\,g:$$

As before we have

$$A_i^m = \frac{2}{\pi}\int_o^\pi \sin m\,\varepsilon \sin i\,g\,.\,d\,g; \quad B_i^m = \frac{2}{\pi}\int_o^\pi \cos m\,\varepsilon \cos i\,g\,.\,d\,g.$$

Integrating by parts and omitting terms that are zero at the limits we have

$$A_i^m = \frac{2\,m}{i\,\pi}\int_o^\pi \cos i\,g \cos m\,\varepsilon\,.\,d\,\varepsilon; \; B_i^m = \frac{2\,m}{i\,\pi}\int_o^\pi \sin i\,g \sin m\,\varepsilon\,.\,d\,\varepsilon.$$

Putting for g its value, $\varepsilon - e \sin \varepsilon$, and then changing the products of the cosines and sines to sums and differences we find

$$A_i^m = \frac{m}{i\,\pi}\int_o^\pi \cos\left[(i+m)\,\varepsilon - i\,e \sin \varepsilon\right].\,d\,\varepsilon + \frac{m}{i\,\pi}\int_o^\pi \cos\left[(i-m)\,\varepsilon - i\,e\sin\varepsilon\right].\,d\,\varepsilon,$$

$$B_i^m = \frac{m}{i\,\pi}\int_o^\pi \cos\left[(i-m)\,\varepsilon - i\,e \sin \varepsilon\right].\,d\,\varepsilon - \frac{m}{i\,\pi}\int_o^\pi \cos\left[(i+m)\,\varepsilon - i\,e\sin\varepsilon\right].\,d\,\varepsilon.$$

These integrals are of the same form as that in equation (2). Hence if we put

$$J_k^i = \frac{1}{\pi}\int_o^\pi \cos\,(i\,\varepsilon - k \sin \varepsilon)\,.\,d\,\varepsilon \quad \ldots\ldots\ldots\ldots.(3)$$

we have

$$A_i^m = \frac{m}{i}\,(J_{i\,e}^{i+m} + J_{i\,e}^{i-m}); \quad B_i^m = \frac{m}{i}\,(J_{i\,e}^{i-m} - J_{i\,e}^{i+m}).$$

If therefore we have a table of the J function we can compute the coefficients easily. This function occurs also in the solution of the partial differential equations which are found in the theories of wave motion and of heat; and as it was first investigated and tabulated by Bessel it is called by the German mathematicians the Besselian function. More complete tables have been computed by Hansen, who has investigated this

function in his peculiar way, and who has brought out many curious properties.

We can find an equation between three successive values of the function as follows: Let

$$u = \sin (i\,\varepsilon - k \sin \varepsilon),$$

then

$$d\,u = i \cos (i\,\varepsilon - k \sin \varepsilon)\, d\,\varepsilon - \frac{k}{2} \cos [(i+1)\,\varepsilon - k \sin \varepsilon]\, d\,\varepsilon$$
$$- \frac{k}{2} \cos [(i-1)\,\varepsilon - k \sin \varepsilon]\,.\,d\,\varepsilon.$$

Since u is zero at the limits o and π we have by integrating this value of $d\,u$,

$$k\,J_k^{i-1} - 2\,i\,J_k^i + k\,J_k^{i+1} = 0. \quad \ldots\ldots\ldots\ldots\ldots (4)$$

This equation gives the value of the function from the values of the two lower orders, but it is not well adapted to numerical calculation, since it gives the value of a small quantity from the difference of two greater ones. It can however be easily transformed into a continued fraction well suited to such calculation.

From equation (3) we have

$$J_k^i = \frac{1}{\pi}\int_o^\pi \cos i\,\varepsilon \cos (k \sin \varepsilon)\,.\,d\,\varepsilon + \frac{1}{\pi}\int_o^\pi \sin i\,\varepsilon \sin (k \sin \varepsilon)\,.\,d\,\varepsilon.$$

And hence

$$J_k^o = \frac{1}{\pi}\int_o^\pi \cos (k \sin \varepsilon)\,.\,d\,\varepsilon = \frac{1}{\pi}\int_o^\pi d\,\varepsilon$$
$$\times \left\{ 1 - \frac{k^2 \sin^2 \varepsilon}{1\,.\,2} + \frac{k^4 \sin^4 \varepsilon}{1.2.3.4} - \frac{k^6 \sin^6 \varepsilon}{1.2.3.4.5.6} + \ldots \right\},$$

$$J_k^1 = \frac{1}{\pi}\int_o^\pi \sin \varepsilon \sin (k \sin \varepsilon)\,.\,d\,\varepsilon = \frac{1}{\pi}\int_o^\pi d\,\varepsilon$$
$$\times \left\{ k \sin^2 \varepsilon - \frac{k^3 \sin^4 \varepsilon}{1.2.3} + \frac{k^5 \sin^6 \varepsilon}{1.2.3.4.5} - \ldots \right\}.$$

Integrating the terms of the series between the limits o and π we have

$$J_k^o = 1 - \frac{k^2}{2^2} + \frac{k^4}{(2.4)^2} - \frac{k^6}{(2.4.6)^2} + \frac{k^8}{(2.4.6.8)^2} - \ldots$$

$$J_k^1 = \frac{2k}{2^2} - \frac{4k^3}{(2.4)^2} + \frac{6k^5}{(2.4.6)^2} - \frac{8k^7}{(2.4.6.8)^2} + \ldots$$

These values and equation (4) enable us to compute all values of the function. An elegant derivation of the general series for J_k^i will be found in Schlömilch's Compendium of the Higher Analysis, Vol. 2, p. 157.

This derivation depends on the following formula given by Jacobi, (Crelle, Vol. 15):

$$\int_0^\pi \phi(\cos t) \cos n\, t \,.\, d\, t = \frac{1}{1.3.5\ldots(2\, n-1)} \,.\, \int_0^\pi \phi^{(n)} (\cos t) \sin^{2n} t \,.\, d\, t,$$

where (n) denotes repeated differentiations.

DISCUSSION OF AN EXPONENTIAL CURVE.

BY IRVING P. CHURCH, B. C. E., NEWBURGH, N. Y.

Construction by Points.—Let $O\ C = c$ be the base of any logarithmic system. On $O\ U$ and $C\ V$, perpendicular to $O\ C$, lay off $O\ D = u$, and $C\ R = v$, such that $u = \log v$ in the system whose base is c. The intersection of $O\ R$ and $D\ P$, (parallel to $O\ C$) fixes P, a point of the curve.

From the construction we can immediately derive the polar equation of the curve:

$$O\ P = s = \frac{u}{\sin \phi} = \frac{\log v}{\sin \phi} = \frac{\log .\, [c \tan .\, \phi]}{\sin \phi}$$
$$= \frac{\log .\, c + \log \tan \phi}{\sin \phi} \quad \ldots\ldots\ldots\ldots (1)$$

Substituting from $y = s \sin \phi$, and $\tan \phi = \frac{y}{x}$, we have for rectangular co-ordinates,

$$y = \log .\, c + \log \left(\frac{y}{x}\right) = \log .\, \left\{\frac{cy}{x}\right\} \text{ or } c^y = \frac{cy}{x},$$

whence we have the final equation of the curve,

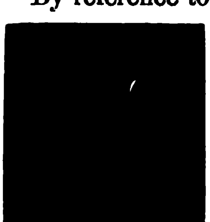
$$x = \frac{y}{c^{y-1}}; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

and also successively,

$$\frac{dx}{dy} = c^{1-y} (1 - y\, l\, c),$$

$$\text{and } \frac{d^2 x}{d y^2} = c^{1-y}\, l\, c\, (y\, l\, c - 2);\ \text{here } l\, c = \text{Nap. log} .\, c.$$

By reference to the equation (2) we see that if $c >$ unity the curve has the form indicated by our first diagram, which will give a maximum for x; also if $c =$ unity, (2) becomes $x = y$, the equation of a straight line bisecting the first angle, while if $c <$ unity, the curve passes to the other side of the line just mentioned, and is tangent to Y at $y = -\infty$, thus, and gives a minimum for x.



It is now our object to find, when $c > 1$, its particular value, such that $c = x$ max. From

$$\frac{dx}{dy} = 0 \text{ we have } y = \frac{1}{l\,c}, \text{ whence } x \text{ max.} = \frac{\frac{1}{l\,c}}{c^{\left(\frac{1}{l\,c} - 1\right)}},$$

and putting this equal to c, we have

$$\frac{1}{l\,c} = c^{\left(\frac{1}{l\,c}\right)}, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

or $M = (\text{base})^M$. (3), solved by successive approximations I find to give, to six decimals

$$c = 1.444667\ldots. = x \text{ max.},$$

for which value

$$y = \frac{1}{l\,c} = 2.\,71828\ldots. = e = \text{Nap. base.}$$

Now by reference to the first figure and construction of this article, we easily perceive that, if $C\ V$ cut the curve, then for any such point of intersection we must have $u = v$ or $v = \log v$ in the system whose basis is c.

Moreover, when $c = x$ max., $C\ V$ is tangent to the curve, giving

$$u = v = y = e = c^e = 1.444667^e\ldots.$$

Also noticing that if $c > 1.444667\ldots.$ $C\ V$ cannot intersect the curve; that for

$$c \begin{matrix} > 1 \\ < 1.444667\ldots. \end{matrix}$$

$C\ V$ intersects it in two points; and that for

$$c \begin{matrix} = \\ < \end{matrix} 1$$

$C\ V$ intersects it but once, we are prepared to make the following deductions:

I. *In logarithmic systems with bases greater than* $e^{\frac{1}{e}} = 1.444667\ldots.$ *there can be no logarithm equal to its antilogarithm or natural number.*

II. *In the system whose base is* $e^{\frac{1}{e}} = 1.444667\ldots.,$

$$\textit{if } e = \frac{1}{l\,c} \textit{ then } l\,c = \frac{1}{e};\ \textit{i. e. } c = e^{\frac{1}{e}},$$

there is but one such logarithm and its value is $2.718\ldots. = e =$ *Nap. base, and it is at the same time the modulus* (M) *of the system,*

$$\textit{for } e = \frac{1}{l\ (1.444667\ldots.)}.$$

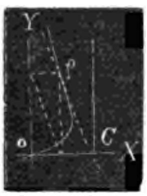
III. ***In systems whose bases are less than unity, there is but one such logarithm and it is less than unity.***

Point of Inflection.—If we put

$$\frac{d^2 x}{d y^2} = 0 \text{ we have } y = \frac{2}{l c},$$

which substituted in

$$\frac{dx}{dy} \text{ gives } \frac{dx}{dy} = \tan \omega = -c^{1-\frac{2}{l}};$$



also in the value of $x = \frac{y}{c^{y-c}}$ and we have $-\frac{x}{y} = -c^{1-\frac{2}{l}}$.

$$\therefore \tan \omega = \left(-\frac{x}{y}\right)$$

for point of inflection. That is, ***the tangent to the curve at the point of inflection is parallel to a diagonal of the rectangle formed by the co-ordinates of the point of inflection and the co-ordinate axes.***

SOLUTIONS OF PROBLEMS IN NO. 3.

Solutions have been received as follows: S.J. Child solved 12; Theo. L. DeLand solved 11; Prof. A. B. Evans solved 11, 12, 13 and 14; Prof. J. M. Greenwood solved 13 and 14; Henry Gunder solved 11 and 12; William Hoover solved 12; Artemas Martin solved 11, 12, 13 and 14; L, Regan solved 12; Walter Siverly solved 11, 12, 13 and 14; S. W. Salmon solved 11 and 12; Prof. J. Scheffer solved 11 and 12; and E. B. Seitz solved 15.

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11. "Borrowed a sum of money at 8 per cent. simple interest and loaned it out again at 5 per cent. compound interest; in what time will I gain the amount borrowed."

SOLUTION BY HENRY GUNDER, GREENVILLE, OHIO.

Let the sum borrowed be \$1, and put t for the time to *gain* this sum. Then by the conditions we have $(1.05)^t = 2 + .08\,t$. By trial we find $t = 30$ years $+ x$, a fraction of a year. To find this fraction we have the following equation: $(1.05)^{30} + (1.05)^{30} \times .05\,x = 4.40 + .08\,x$, from which we find $x = .5735$ years; hence the time is 30y. 6m. 26d.

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12. "Given the base $A\,C$ of a triangle and the ratio of $A\,B$ to $B\,C$ to find the locus of the point B by Geometry."

SOLUTION BY ARTEMAS MARTIN, ERIE, PA.

Put $AC = a$ and the ratio of AB to BC as m to n; then

$$(AB)^2 : (BC)^2 :: m^2 : n^2.$$

Take the origin at A and let $AD = x$, $BD = y$; then $AB = \sqrt{x^2 + y^2}$ and $BC = \sqrt{(a-x)^2 + y^2}$;

$$\therefore x^2 + y^2 : (a-x)^2 + y^2 :: m^2 : n^2;$$

$$\text{whence } \left(x - \frac{m^2 a}{m^2 - n^2}\right)^2 + y^2 = \left(\frac{mna}{m^2 - n^2}\right)^2,$$

the equation to a circle whose radius is

$$\frac{mna}{m^2 - n^2}.$$

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13. "A body is projected at a given distance from the center of force with a given velocity, and in a direction perpendicular to that distance: When the force is repulsive and varies inversely as the cube of the distance, find the path of the body."

SOLUTION BY ASHER B. EVANS, LOCKPORT, N. Y.

Let r and θ be the angular co-ordinates of the body at the time t from the commencement of motion, the origin being the center of force. Let v be the velocity of projection, and let the prime radius be so taken that $r = a$, and $\theta = 0$ at the instant of projection.

From the usual equations of motion we have, μ being the intensity of the repulsive force at a unit's distance,

$$v^2 = r^2\left(\frac{d\theta}{dt}\right)^2 + \left(\frac{dr}{dt}\right)^2 = v_1^2 + \int_a^r \frac{2\mu\, dr}{r^3} = v_1^2 + \mu\left(\frac{1}{a^2} - \frac{1}{r^2}\right)$$

$$\text{and } r^2\left(\frac{d\theta}{dt}\right) = c.$$

Eliminating dt between these equations, we find for the differential equation of the curve

$$\frac{c^2}{r^4}\left(\frac{dr}{d\theta}\right)^2 = v_1^2 + \frac{\mu}{a^2} - \frac{c^2 + \mu}{r^2}. \quad \ldots\ldots\ldots\ldots\ldots (1)$$

$$\text{Put } m^2 \text{ for } \frac{c^2 + \mu}{v_1^2 + \frac{\mu}{a^2}}$$

and reduce (1) to the form

$$\frac{-c^2}{c^2 + \mu}\left\{d\left(\frac{m}{r}\right)\right\}^2 = \left\{1 - \left(\frac{m}{r}\right)^2\right\} d\theta^2;$$

$$\text{or} \quad \frac{d\left(\frac{m}{r}\right)}{\sqrt{1-\left(\frac{m}{r}\right)^2}} = \frac{-\sqrt{c^2+\mu}}{c}\, d\,\theta. \quad \ldots\ldots\ldots\ldots (2)$$

Integrating (2) we find

$$c\left\{\sin^{-1}\left(\frac{m}{a}\right) - \sin^{-1}\left(\frac{m}{r}\right)\right\} = \theta\sqrt{c^2+\mu}$$

for the equation of the required path.

PROBLEMS.

20. It is required to circumscribe about a given parabola an isosceles triangle whose area shall be a minimum.—Communicated by A. W. Mason, Audenreid, Carbon County, Pa.

21. $A\,B\,C\,D$ is a quadrilateral; O, the intersection of the diagonals; P, Q, points in $B\,D$, $A\,C$, such that $Q\,A = O\,C$ and $P\,B = O\,D$. Prove that the center of gravity of the quadrilateral coincides with that of the triangle $O\,P\,Q$.—Communicated by I. H. Turrell, Cumminsville, Ohio.

22. Show that the distance from a vertex of any plane triangle to the points where the opposite escribed circle touches the sides meeting at that vertex is constant and equal to half the sum of the sides of the triangle.—Communicated by Prof. C. M. Woodward, St. Louis, Mo.

23. If the brightness of the moon be equal to the brightness of the clouds by day, show that the light of an overcast day is to that of a full moon-lit night as $8(360)^2 : \pi^2$; the diameter of the moon being $30'$.—Communicated by Prof. Jas. Nooney, New Haven, Conn.

24. There are m labels, to be distributed by lot among m different articles. Required the probable amount of coincidence in two independent allotments.—Communicated by Prof. Pliny Earle Chase, Haverford College, Pa.

EDITORIAL NOTES.

We have been obliged, for want of room, to defer publishing the solutions of Nos. 14 and 15 to our next issue. We regret that our limited space will not permit us to publish the elegant solution of No. 12 by Prof. Scheffer, and of No. 18 by Messrs. Greenwood and Martin.

We publish in this number a translation of a letter received from Prof. Schiaparelli in reply to Prof. Hall's article, on Comets and Meteors, published in No. 2, which we think will interest most of our readers. In a note to the editor, Prof. Hall writes: "It is not difficult I think to reply to the two propositions laid down by Prof. Schiaparelli, but as he fully concedes that the solution given by Laplace is practically correct, I have nothing more to say."

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THE ANALYST.

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THE POTENTIAL FUNCTION.

BY PROF. D. TROWBRIDGE, WATERBURGH, N. Y.

1. There is a function of constant occurrence in all investigations that relate to attractions, and it is of so much importance that I purpose to give, in this paper, a brief explanation of its nature, composition and use. Several terms have been proposed to express its nature, and by which it may be known. It has been proposed to call it the *Integral Proximity* of the attracting mass; Mr. Green and Prof. MacCullagh called it the *Potential* of the attracting mass; and Dr. Whewell proposed to call it the *Integral Potentiality*, or *Integral Attractivity* (*Hist. Inductive Sciences*, Vol. III, p. 76). The term by which it is now generally known, however, is *Potential*. It would seem that the term was chosen because the function has some relation to the *power* of the body, while the actual force exerted by the mass results at once from the function, by differentiation. The potential function is often called V.

2. If m be the mass of a particle, and r its distance from an attracted point, then, according to Newton's law of gravity, the force of attraction of the mass m, at the distance r and along the line r, is

$$\frac{m}{r^2}.\quad \text{If we put } V=\frac{m}{r},\ \text{then}-D_r V=\frac{m}{r^2}.$$

If m be the differential of a mass M, then V will be the sum of all the particles of M, divided by their distances from the attracted point. The integral calculus will enables us to find the value of this sum. Since $m = d\,M$, we shall have

$$V=\int\frac{d\,M}{r}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

the integral extending over the whole attracting mass.

Let x, y, z be the rectangular coördinates of any point of the body M,

a, b, c the rectangular coördinates of the attracted point; then if ρ be the density of M at the point $x\ y\ z$, we shall have

$$d\,M = \rho\ dx\ d\,y\ d\,z,\ r^2 = (a-x)^2 + (b-y)^2 + (c-z)^2 \ldots\ldots (2)$$

and

$$V = \iiint \frac{\rho\ d\,x\ d\,y\ d\,z}{\sqrt{(a-x)^2 + (b-y)^2 + (c-z)^2}}, \ldots\ldots\ldots\ldots (3)$$

the limits of x, y and z being derived from the equation of the surface of the attracting body. In equation (3) a, b, c are entirely independent of ρ, x, y, z, and if we let X, Y and Z be the force of attraction resolved along the coördinate axes x, y, z respectively, we shall have

$$-D_a V = X = -\iiint \frac{\rho\,(a-x)d\,x\ d\,y\ d\,z}{[(a-x)^2 + (b-y)^2 + (c-z)^2]^{\frac{3}{2}}},$$

$$-D_b V = Y = -\iiint \frac{\rho\,(b-y)d\,x\ d\,y\ d\,z}{[(a-x)^2 + (b-y)^2 + (c-z)^2)]^{\frac{3}{2}}},$$

$$-D_c V = Z = -\iiint \frac{\rho\,(c-z)d\,x\ d\,y\ d\,z}{[(a-x)^2 + (b-y)^2 + (c-z)^2]^{\frac{3}{2}}} \ldots\ldots (4)$$

From these equations we easily find

$$\int (D_a V d\,a + D_b V d\,b + D_c V d\,c) = V = -\int (X d\,a + Y d\,b + Z d\,c) + C.(5)$$

The reader will understand that $D_a V$ is a partial differential coefficient of V with respect to a, on the supposition that V and a are the only varible quantities in equation (3). The same is true of $D_b V$ and $D_c V$. The subject of partial differential coefficients should be thoroughly understood by the student. In some cases the variation of the quantities is entirely arbitrary, as in equations (4), and it is made purely for analytical convenience; and in others the variation is partial, but *may* exist independent of other quantities.

3. The potential must be calculated for the point which we suppose attracted by the mass whose potential we calculate. Thus, we may suppose the potential calculated for *any* point $a\ b\ c$, *external* to the attracting mass, or for *any* point $a\ b\ c$, *internal* to the attracting mass, but the calculation will be different for the two cases.

Except for a few cases the calculation of the potential seems to be one of extreme difficulty, and even beyond the present resources of analysis, if the complete expression be required. Laplace devised a method, however, for resolving the integral, which gives the potential, into a series. The terms of this series possess some very important properties which are very useful in the higher parts of mathematical physics. Certain co-

efficients in this series are technically called *Laplace's Coefficients.* It is not my purpose to discuss, in this paper, the properties of these coefficients. The discussion is usually thought to be very difficult; and as it is usually presented, it is, but it is not necessarily so.

4. Now suppose the value of V found; it will be a function of the coördinates a, b, c. Suppose we wish to transform the coördinates from a, b, c to a', b', c'. Let $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$, and $\alpha'', \beta'', \gamma''$, be the angles made by a, b, c with the axes of a', b', c', respectively. Then we shall have (see Davies' *Analytical Geometry*, Book IX,)

$$a = a' \cos \alpha + b' \cos \beta + c' \cos \gamma,\ b = a' \cos \alpha' + b' \cos \beta' + c' \cos \gamma',$$
$$c = a' \cos \alpha'' + b' \cos \beta'' + c' \cos \gamma'' \ldots\ldots\ldots\ldots\ldots (6)$$

Since we now suppose V to be a function of a', b', c', we shall have

$$D_a' V = D_a V D_a' a + D_b V D_a' b + D_c V D_a' c$$
$$= - [X D_a' a + Y D_a' b + Z D_a' c] \ldots\ldots\ldots\ldots (7)$$

From equation (6) we have

$$D_a' a = \cos \alpha \ D_a' b = \cos \alpha',\ D_a' c = \cos \alpha'';$$

and these substituted in equation (7) give

$$D_a' V = - [X \cos \alpha + Y \cos \alpha' + Z \cos \alpha''] = - X',$$

the force in the direction of the axis of a' (as Analytical Geometry wil show). In a precisely similar manner we could find

$$D_b' V = - Y', \text{ and } D_c' V = - Z'.$$

We hence conclude that, if we have V calculated for *any* axes whatever we can find the components of the attractive force in the direction of any other axes, by substituting for the original co-ordinates their values in terms of the new ones, and then taking the partial differential coefficients with respect to the new coördinates.

5. If we substitute polar coördinates in equation (3), we shall have, if we make r, θ, ω the polar coördinates of the attracted point, and, r', θ', ω' the polar coördinates of any point of the solid,

$$a = r \sin \theta \cos \omega,\ b = r \sin \theta \sin \omega,\ c = r \cos \theta \ldots\ldots\ldots (8)$$
$$x = r' \sin \theta' \cos \omega',\ y = r \sin \theta' \cos \omega',\ z = r' \cos \theta' \ldots\ldots (9)$$
$$r^2 = a^2 + b^2 + c^2,\ r'^2 = x^2 + y^2 + z^2 \ldots\ldots\ldots\ldots (10)$$
$$(a-x)^2 + (b-y)^2 + (c-z)^2 = r^2 + r'^2 - 2rr'[\cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\omega - \omega')]$$
$$d x\, d y\, d z = r'^2 \sin \theta'\, d r'\, d \theta'\, d \omega'.$$

With these values we have

$$V=\iiint\frac{\rho\, r'^2 \sin\theta'\, d\,r'\, d\,\theta'\, d\,\omega'}{[r^2+r'^2-2\,rr'(\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\omega-\omega')]^{\frac{1}{2}}}\,.(11)$$

This equation is very much simplified if we place the origin of coördinates at the attracted point, for then $r=0$, and equation (11) becomes

$$V=\iiint\rho\, r'\sin\theta'\, d\,r'\, d\,\theta'\, d\,\omega'\ldots\ldots\ldots\ldots(12)$$

From this equation we see that the value of V is continuous, or does not become infinite as appears possible from equation (11).

6. Thus far we have considered V a continuous function, and made up of the sum of all the particles of the attracting mass, divided by their respective distances from the attracted point. There is another case in which we may consider the attracting particles (or masses) as independent and finite in number. Thus, if we wish to find the potential of the masses m', m'', &c., for their attractive influence on the mass m, the distances being r', r'', &c., we should have

$$V=+\frac{m'}{r'}\ \frac{m''}{r''}+\frac{m'''}{r'''}+\ldots=\Sigma\ \left(\frac{m'}{r'}\right),\text{ suppose}\ldots\ldots(13)$$

When we wish to find the potential for the *disturbing force* exerted by m', m'', &c., on the motion of m, the form of the expression becomes changed, and it is usually called R, the *Disturbing Function.* This form of V is employed in the Lunar and the Planetary Theory.

7. We shall now show how to find V for a sphere of uniform density. If we use equation (12) we must integrate with respect to ω' between the limits 0 and $2\,\pi$; r', between the limits r'_1 and r'_2 for an external point; and between the limits r_1' and 0 for an internal point. If we put $\mu=\cos\theta'$ we shall have

$$V=\pi\,\rho\int(r'^2_1-r'^2_2)d\,\mu\ldots(14)\quad\text{and}\quad V=\pi\,\rho\int(r'^2_1\,d\,\mu\ldots(15)$$

The equation of the generating circle of the sphere is

$$x^2+y^2=a^2\ldots\ldots\ldots\ldots\ldots\ldots(16)$$

Put $c=$ the distance between the attracted point and the center of the sphere, and take this line for the axis of x. We then have

$$x=r'\cos\theta'-c,\ y=r'\sin\theta'.$$

These values in (16) give

$$r'^2-2\,c\,r'\cos\theta'=a^2-c^2,\ \therefore\ r'=c\,\mu\pm\sqrt{a^2-c^2+c^2\,\mu^2}.$$

These two values of r' give

$$r_1' = c\,\mu + \sqrt{a^2 - c^2 + c^2\,\mu^2},\ r'_2 = c\,\mu - \sqrt{a^2 - c^2 + c^2\,\mu^2}.$$

$$r'^2_1 - r'^2_2 = 4\,c\,\mu\sqrt{a^2 - c^2 + c^2\,\mu^2}$$

$$r'^2_1 = 2\,c^2\,\mu^2 + a^2 - c^2 + 2\,c\,\mu\sqrt{a^2 - c^2 + c^2\,\mu^2}.$$

The limits of μ for an external point are $\mu = +1$, and $c^2\,\mu^2 = c^2 - a^2$; and for an internal point the limits of μ are $+1$ and -1. The equation (14) now gives, if $c^2\,\mu'^2 = c^2 - a^2$,

$$V = 4\,\pi\,\rho\int_{+\mu'}^{+1} c\,\mu\,d\,\mu\sqrt{a^2 - c^2 + c^2\,\mu^2} = \frac{4\,\pi\,\rho\,a^3}{3\,c} = \frac{\text{mass}}{c}.$$

Equation (15) gives

$$V = 2\,\pi\,\rho\int_{-1}^{+1} d\,\mu\,(2\,c^2\,\mu^2 + a^2 - c^2 + 2\,c\,\mu\sqrt{a^2 - c^2 + c^2\,\mu^2}) = 2\,\pi\,\rho\,a^2 - \tfrac{2}{3}\,\pi\,\rho\,c^2.$$

We can find the potential of a homogeneous spheroid of revolution, for a point situated in the axis of revolution, by a similar process, though the work is more complicated. Since the values of X, Y, Z can be calculated directly, we can find V for the spheroid by equation (5), the value of C being made the value of V for the center of the spheroid.

SOLUTION OF TWO SIMILAR INDETERMINATE PROBLEMS.

BY GEORGE R. PERKINS, LL. D., UTICA, N. Y.

1. *Find three square numbers in arithmetical progression, such that if from each its root be subtracted, the remainders shall be squares.*

2. *Find three square numbers in arithmetical progression, such that if to each its root be added, the sums shall be squares.*

We will assume for the three square numbers as follows:

$$\{\tfrac{1}{4}(x+x^{-1})\pm\tfrac{1}{2}\}^2;\ \{\tfrac{1}{4}(y+y^{-1})\pm\tfrac{1}{2}\}^2;\ \{\tfrac{1}{4}(z+z^{-1})\pm\tfrac{1}{2}\}^2.$$

The *upper* sign corresponds with the first problem, and the *lower* sign with the second problem.

Since the following condition is true for all values of x,

$$\{\tfrac{1}{4}(x+x^{-1})\pm\tfrac{1}{2}\}^2 \mp \{\tfrac{1}{4}(x+x^{-1})\pm\tfrac{1}{2}\} = \{\tfrac{1}{4}(x-x^{-1})\}^2, \ldots\ldots(1)$$

we see that the square numbers as above assumed, will satisfy the con-

ditions of the problems as to the square of the remainders in the *first*, and the square of the sums in the *second*.

It only remains therefore to determine x, y and z so that these assumed numbers shall be in arithmetical progression. That is, we must have

$$\left.\begin{array}{l}\{\tfrac{1}{2}(y+y^{-1})\pm\tfrac{1}{2}\}^2-\{\tfrac{1}{2}(x+x^{-1})\pm\tfrac{1}{2}\}^2= \\ \{\tfrac{1}{2}(z+z^{-1})\pm\tfrac{1}{2}\}^2-\{\tfrac{1}{2}(y+y^{-1})\pm\tfrac{1}{2}\}^2\end{array}\right\}\ldots\ldots\ldots\ldots(2)$$

Since the difference of the squares of two quantities is equal to the product of their sum into their difference, we must have

$$\left.\begin{array}{l}\{\tfrac{1}{2}(y+y^{-1})+\tfrac{1}{2}(x+x^{-1})\pm1\}\times\{\tfrac{1}{2}(y+y^{-1})-\tfrac{1}{2}(x+x^{-1})\}= \\ \{\tfrac{1}{2}(z+z^{-1})+\tfrac{1}{2}(y+y^{-1})\pm1\}\times\{\tfrac{1}{2}(z+z^{-1})-\tfrac{1}{2}(y+y^{-1})\}\end{array}\right\}\ldots(3)$$

Condition (3) will be satisfied by the following assumed values:

$$\tfrac{1}{2}(y+y^{-1})+\tfrac{1}{2}(x+x^{-1})\pm1=m\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

$$\tfrac{1}{2}(y+y^{-1})-\tfrac{1}{2}(x+x^{-1})\qquad=n\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

$$\tfrac{1}{2}(z+z^{-1})+\tfrac{1}{2}(y+y^{-1})\pm1=\frac{(p+1)m}{p}\ldots\ldots\ldots\ldots(6)$$

$$\tfrac{1}{2}(z+z^{-1})-\tfrac{1}{2}(y+y^{-1})\qquad=\frac{pn}{p+1}\ldots\ldots\ldots\ldots(7)$$

Conditions (4) and (5) give

$$\tfrac{1}{2}(y+y^{-1})\pm1=m+n\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

$$\tfrac{1}{2}(x+x^{-1})\pm1=m-n\ldots\ldots\ldots\ldots\ldots\ldots(9)$$

Conditions (6) and (7) give

$$\tfrac{1}{2}(z+z^{-1})\pm1=\frac{(p+1)\,m}{p}+\frac{p\,n}{p+1}\ldots\ldots\ldots\ldots(10)$$

$$\tfrac{1}{2}(y+y^{-1})\pm1=\frac{(p+1)\,m}{p}-\frac{p\,n}{p+1}\ldots\ldots\ldots\ldots(11)$$

Putting the values of $\tfrac{1}{2}(y+y^{-1})\pm1$, given by (8) and (11), equal, we find

$$n=\frac{(p+1)\,m}{p(2\,p+1)}\ldots\ldots\ldots\ldots\ldots\ldots(12)$$

Hence, we have

$$\tfrac{1}{2}(x+x^{-1})\pm1=\frac{(2\,p^2-1)\,m}{p(2\,p+1)}\ldots\ldots\ldots\ldots(13)$$

$$\tfrac{1}{2}(y+y^{-1})\pm1=\frac{(2\,p^2+2\,p+1)\,m}{p(2\,p+1)}\ldots\ldots\ldots\ldots(14)$$

$$\tfrac{1}{2}(z + z^{-1}) \pm 1 = \frac{(2p^2 + 4p + 1)\,m}{p(2p+1)} \dots\dots\dots\dots (15)$$

Putting $a = \dfrac{2p(2p+1)}{2p^2 - 1}$; $b = \dfrac{2p(2p+1)}{2p^2 + 2p + 1}$; $c = \dfrac{2p(2p+1)}{2p^2 + 4p + 1}$,

and solving (13), (14) and (15), we obtain

$$x = \mp 1 + \frac{2}{a}\left\{ m + \sqrt{m^2 \mp a\,m} \right\} \dots\dots\dots\dots (16)$$

$$y = \mp 1 + \frac{2}{b}\left\{ m + \sqrt{m^2 \mp b\,m} \right\} \dots\dots\dots\dots (17)$$

$$z = \mp 1 + \frac{2}{c}\left\{ m + \sqrt{m^2 \mp c\,m} \right\} \dots\dots\dots\dots (18)$$

We must now determine m so as to make these values of x, y and z rational, that is, we must have

$$m^2 \mp a\,m = \square;\; m^2 \mp b\,m = \square;\; m^2 \mp c\,m = \square \dots\dots (a)$$

If we assume $m^2 + a\,m = (m + k)^2$, we shall find

$$m = \frac{k^2}{\pm(2k - a)} \dots\dots\dots\dots\dots\dots\dots\dots (19)$$

This causes the other conditions of (a) to become

$$\left.\begin{aligned} m^2 \mp b\,m &= \frac{k^2\{k^2 - 2bk + ab\}}{(2k-a)^2} \\ m^2 \mp c\,m &= \frac{k^2\{k^2 - 2ck + ac\}}{(2k-a)^2} \end{aligned}\right\} \dots\dots\dots\dots (b)$$

Hence, we must have

$$\left.\begin{aligned} k^2 - 2bk + ab &= \square \\ k^2 - 2ck + ac &= \square \end{aligned}\right\} \dots\dots\dots\dots\dots\dots (c)$$

Assume, $k^2 - 2bk + ab = (k - l)^2$, and we obtain

$$k = \frac{l^2 - ab}{2(l - b)} \dots\dots\dots\dots\dots\dots\dots\dots (20)$$

And the second condition (c) becomes

$$k^2 - 2ck + ac = \frac{l^4 - 4cl^3 + 2(2ac + 2bc - ab)l^2 - 4abcl + a^2b^2}{4(l-b)^2}.$$

Consequently,

$$l^4 - 4cl^3 + 2(2ac + 2bc - ab)l^2 - 4abcl + a^2b^2 = \square \dots (21)$$

Assuming as follows:

$$l^4 - 4cl^3 + 2(2ac + 2bc - ab)l^2 - 4abcl + a^2b^2$$

$$=\left\{ l^2 - 2\,c\,l + 2\,c(a+b-c) - a\,b \right\}^2, \text{ we find}$$

$$l = \tfrac{1}{2}(a+b-c) \ldots\ldots\ldots\ldots\ldots\ldots \quad (22)$$

This value substituted in (20) gives

$$k = \frac{2(a\,b + a\,c + b\,c) - (a^2 + b^2 + c^2)}{4(-a+b+c)} \ldots\ldots\ldots \quad (23)$$

And substituting this value of k in (19) we find

$$m = \frac{\left\{ 2(a\,b + a\,c + b\,c) - (a^2 + b^2 + c^2 \right\}^2}{\pm 8(-a+b+c).(a-b+c).(a+b-c)} \ldots \quad (24)$$

Conditions (13), (14) and (15) give

$$\tfrac{1}{4}(x + x^{-1}) \pm \tfrac{1}{2} = \frac{m}{a} \ldots\ldots\ldots\ldots \quad (25)$$

$$\tfrac{1}{4}(y + y^{-1}) \pm \tfrac{1}{2} = \frac{m}{b} \ldots\ldots\ldots\ldots \quad (26)$$

$$\tfrac{1}{4}(z + z^{-1}) \pm \tfrac{1}{2} = \frac{m}{c} \ldots\ldots\ldots\ldots \quad (27)$$

Hence, the three square numbers sought are

$$\left(\frac{m}{a}\right)^2; \quad \left(\frac{m}{b}\right)^2; \quad \left(\frac{m}{c}\right)^2.$$

These roots in terms of p are as follows:

$$\frac{m}{a} = \frac{(2\,p^2 - 1)A^2}{\pm 8(2\,p^2 - 1).(2\,p^2 + 2\,p + 1).(2\,p^2 + 4\,p + 1)\,B\,C\,D} \ldots \quad (31)$$

$$\frac{m}{b} = \frac{(2\,p^2 + 2\,p + 1)A^2}{\pm 8(2\,p^2 - 1).(2\,p^2 + 2\,p + 1).(2\,p^2 + 4\,p + 1)\,B\,C\,D} \ldots \quad (32)$$

$$\frac{m}{c} = \frac{(2\,p^2 + 4\,p + 1)A^2}{\pm 8(2\,p^2 - 1).(2\,p^2 + 2\,p + 1).(2\,p^2 + 4\,p + 1)\,B\,C\,D} \ldots \quad (33)$$

In the above, we have

$A = 48\,p^8 + 192\,p^7 + 224\,p^6 - 280\,p^4 - 336\,p^3 - 184\,p^2 - 48\,p - 5.$

$B = 4\,p^4 - 12\,p^2 - 12\,p - 3.$

$C = 4\,p^4 + 8\,p^3 + 12\,p^2 + 8\,p + 1.$

$D = 4\,p^4 + 16\,p^3 + 12\,p^2 + 4\,p + 1.$

These results are general and hold good, in the case of the first problem, for all values of p greater than 2.

Taking $p = 3$, we find $A = 864571$; $B = 177$; $C = 673$; $D = 877$;

and these expressions (31), (32) and (33) give for the roots of our numbers, in the case of the first problem, as follows:

$$\frac{m}{a}=\frac{12707211238697}{11011044931800};\ \frac{m}{b}=\frac{18687075351025}{11011044931800};\ \frac{m}{c}=\frac{23171973435271}{11011044931800}$$

In our second problem, the denominator of the expression for m must be negative, and this will be the case for $p=1$, for then we find $A=-389$, $B=-23$, $C=33$, and $D=37$; so that (31), (32) and (33) become

$$\frac{m}{a}=\frac{151321}{7863240};\ \frac{m}{b}=\frac{756605}{7863240};\ \frac{m}{c}=\frac{1059247}{7863240}.$$

If we take $p=-2$, our results will satisfy this second problem, also $p=-3$ will give answers. It will also be satisfied for $p=2$.

Remark.—On page 494, of Stoddard and Henkle's University Algebra, New York, Edition of 1861, our *first* problem is given. The answer is given by x^2, 25 x^2 and 49 x^2, in very large numbers, consisting of more than three times as many places of figures as in my numbers. The value of the root of the first number is there given

$$x=\frac{2340814414884742932783918468592674193422525281}{20177642715140781960429281969999625135323016 0}.$$

It is stated that these numbers were furnished by Prof. Daniel Kirkwood, and that he believed they were the smallest numbers which could be found. His method of solving the problem is not given.

I do not recollect ever having seen a solution of this *first* problem. In an Elementary Treatise on Algebra by John D. Williams, Boston, 1840, on page 413, he gives the solution of our *second* problem. He also proceeds in his solution by assuming x^2, 25 x^2 and 49 x^2 for the numbers, and obtains for his results the same numbers which I have given.

SOLUTION OF A PROBLEM.

BY PROF. O ROOT, HAMILTON COLLEGE, CLINTON, N. Y.

Problem.—"From a point in the circumference of a circular field a projectile is thrown at random with a given velocity, which is such that the diameter of the field is equal to the greatest range of the projectile; find the chance of its falling within the field."

Solution.—Take the given point as the origin, let the diameter of the circular field be represented by (a); put θ for any angle of elevation and ψ for the angle of azimuth so taken that when the projectile will fall on the circumference of the field we shall have $\psi = 2\theta$. Now since any portion of the surface of a hemisphere whose radius is (a) (the diameter of the given circle) and whose center is at the given point is expressed by

$$a^2 \int \int \cos\theta \, d\theta \, d\psi,$$

therefore the favorable cases will be expressed by the integral

$$a^2 \int_0^{2\theta} d\psi \int_0^{\frac{\pi}{4}} \cos\theta \, d\theta,$$

and this divided by $\dfrac{\pi a^2}{\sqrt{2}}$ will give the chance required; therefore we have

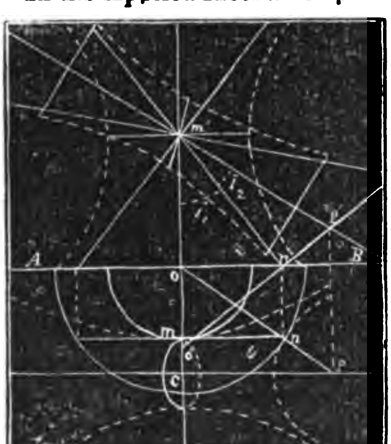
$$\frac{a^2 \int_0^{2\theta} d\psi \int_0^{\frac{\pi}{4}} \cos\theta \, d\theta}{\dfrac{\pi a^2}{\sqrt{2}}} = \frac{1}{2} - \frac{2}{\pi}(\sqrt{2} - 1)$$

tor the chance that the projectile will fall on the circular field.

TANGENCY OF HYPERBOLOIDS OF REVOLUTION.

BY PROF. C. M. WOODWARD, ST. LOUIS, MO.

In his Applied Mechanics p. 430, under the head of skew-bevel wheels, Prof. Rankine says: "If two hyperboloids, equal or unequal be placed in the closest possible contact, they will touch each other along one of the generating straight lines of each which will form their line of contact."



This matter of tangency is stated without proper limitation, but the graphical method given later for finding the obliquities and the gorge circles of the required hyperboloids involves the condition of possibility of such a tangency, which I propose to deduce directly from two tangent hyperboloids by the methods of descriptive geometry.

Let r_1 and r_2 be the radii of the two gorge circles, and i_1 and i_2 the

obliquities of the two surfaces (*i. e.* the angles which the elements make with their respective axes); then the required condition is

$$r_1 : r_2 = \tan i_1 : \tan i_2.$$

Take, as in the Fig., the plane through the axis of the first surface, parallel to the axis of the second as the vertical, and a plane perpendicular to the axis of the first surface, as the horizontal plane. Since the surfaces are tangent, and the plane tangent to both at the point where their gorge circles touch each other contains the common element and is parallel to the vertical plane, the common element is parallel to the vertical plane and its vertical projection is a common asymptote to the vertical projections of the surfaces.

At any point of the element of contact, as at N, (n, n') draw a common normal to the two surfaces. It intersects both axes, one in O, and the other in P.

Since $M\,N$ is parallel to vertical plane, $o'\,n'\,p'$ must be perpendicular to $m'\,n'$. Now the horizontal projecting plane of $M\ N$ divids the perpendicular between the axes into r_1 and r_2, and the normal into $O\ N$ and $N\,P$; hence

$$r_1 : r_2 = O\,N : N\,P$$
$$= o'\,n' : n'\,p'.$$

The obliquities of the surfaces are given in full size in the vertical projection; that is $\quad i_1 = o'\,m'\,n'$ and $i_2 = n'\,m'\,p'$.

Hence, $\quad \tan i_1 : \tan i_2 = o'\,n' : n'\,p'$

and $\quad r_1 : r_2 = \tan i_1 : \tan i_2.$

Q. E. D.

SOLUTIONS OF PROBLEMS IN NOS. 3 AND 4.

Solutions of problems in No. 4, have been received as follows: From Prof. W. E. Arnold, 16 and 18; R. J. Adcock, 17; Marcus Baker, 16, 17, 18 & 19; S. J. Child, 16 & 18; George L. Dake, 16; Prof. A. B. Evans, 16, 17, 18 & 19; Henry Gunder, 16, 17, & 18; Prof. E. W. Hyde, 16, 17, & 18; William Hoover, 16; Phillip Hoglan, 18; Prof. I. N. Jones, 16; J. B. Mott, 16; L. E. Newcomb, 16; A. W. Mason, 16; Prof. A. W. Phillips, 16, 17, 18 & 19; L. Regan, 16; John W. P. Reid, 16; Henry A. Roland, 17 & 19; Miss Kittie Robinson, 16; Prof. Selden Sturges, 16, 17 & 18; E. B. Seitz, 16, 17, 18 & 19; S. W. Salmon, 16, 17, 18 & 19;

R. L. Selden, 16; Walter Siverly, 16, 17, & 19; Elias Schneider, 16; Jas. Stott, 16 & 18; Prof. D. M. Sensenig, 16, 17, 18 & 19; Prof. M. C. Stevens, 16, 17 & 18; and Prof. David Trowbridge, 16, 17, 18 & 19.

14. "Two equal particles attracting each other with forces varying inversely as the square of the distance, are constrained to move in two straight lines at right angles to each other; supposing their motions to commence from rest, to find the time in which each of them will arrive at the intersection of the two straight lines."

SOLUTION BY PROF. J. M. GREENWOOD, KIRKSVILLE, MO.

Take the lines as coördinates, the origin being at their intersection; let a and b represent the distances of the particles from the origin, and x and y their distances at the time t.

From the problem (1) $a^2+b^2=c^2$, (2) $x^2+y^2=r^2$.

The equations of motion are

$$(3)\quad \frac{d^2x}{dt^2}=-\frac{\mu x}{r^3},\quad (4)\quad \frac{d^2y}{dt^2}=-\frac{\mu y}{r^3}.$$

$$(5)\quad \frac{y\,d^2x}{dt^2}-\frac{x\,d^2y}{dt^2}=0\ ;\ \text{also}\ (6)\quad \frac{y\,dx}{dt}=\frac{x\,dy}{dt}.$$

$$(7)\quad \frac{dx}{x}=\frac{dy}{y}\,;\ \therefore\ \log\frac{x}{a}=\log\frac{y}{b}.$$

$$(8)\quad \therefore\ \frac{x}{a}=\frac{y}{b}=\frac{r}{c}.$$

When $x=0$, $y=0$, $r=0$, $\because$ the particles arrive at the origin simultaneously.

To find the time when the force varies inversely as the square of the distance we have (Smith's Mechanics, page 207).

$$(9)\quad dt=\frac{-dr}{\sqrt{2\mu}\sqrt{(\frac{c-r}{cr})}}\qquad \text{Intregating (9)}$$

$$(10)\quad t=\left(\frac{c}{2\mu}\right)^{\frac{1}{2}}\left\{(cr-r^2)^{\frac{1}{2}}-\frac{c}{2}\ \text{versin.}^{-1}\frac{2r}{c}+C\right\}.$$

$$\text{When } t=0,\ r=c;\ \therefore C=\frac{r\pi}{2}=\frac{c\pi}{2}.$$

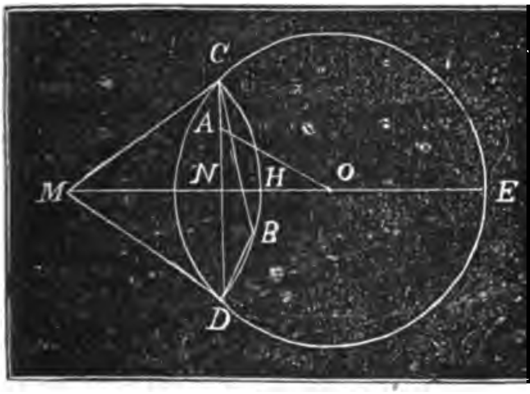
$$(11)\quad t = \left(\frac{c}{2\mu}\right)^{\frac{1}{2}} \left\{ (c\,r - r^2)^{\frac{1}{2}} + \frac{c}{2} \left(\pi - \text{versin}^{-1} \frac{2\,r}{c}\right), \right\}.$$

$$\text{when } r = 0,\ t^2 = \frac{\pi^2 c^3}{8\,\mu}\ ;\ \therefore t = \frac{\pi\, c^{\frac{3}{2}}}{(8\mu)^{\frac{1}{2}}}.$$

15. "Two points are taken at random in the surface of a given circle and a chord drawn through each at random; show that the chance of their intersecting is $\frac{1}{2} + \frac{5}{2\,\pi^2}$."

SOLUTION BY E. B. SEITZ, GREENVILLE, OHIO.

Let $C\,E\,D$ be the given circle, O its center, A, B the random points,



$C\ D$ the random chord through A, $C\,H\,D$ an arc through B, M its center.

Put $O\,A = x$, $N\,H = y$, $C\,N = m$, $O\,G = 1$, $< O\,A\,D = \theta$, $< C\,M\,N = \phi$, area seg. $C\ H\ D = s$, area sector $C\ M\ D = V$, area $\triangle\ C\,M\,D = t$. Then $O\ N = x \sin\,\theta$, $N\,E = 1 + x \sin\theta$, $N\,G = 1 - x \sin\,\theta$, $m = (1 - x^2 \sin^2\theta)^{\frac{1}{2}}$, $C\,M = \frac{m^2 + y^2}{2\,y}$, area seg. $C\,E\,D = \pi - \sin^{-1} m + m\,x \sin\,\theta$, area seg $C\,G\ D = \sin^{-1} m - m\,x \sin\,\theta$, $<$

$$C\,B\,D = \pi - \phi,\ \phi = \sin^{-1}\left(\frac{2\,m\,y}{m^2 + y^2}\right),\ V = \phi\left(\frac{m^2 + y^2}{2\,y}\right)^2,$$

$$t = \frac{m\,(m^2 - y^2)}{2\,y},\ s = V - t.$$

Now when B is in the segment $C\,E\,D$ the limits of y are 0 and $1 + x \sin\,\theta$, of x, 0 and 1, and of θ, 0 and $\frac{1}{2}\,\pi$; and when B is in the segment $C\,G\,D$ the limits of y are 0 and $1 - x \sin\,\theta$, of x, 0 and 1, and of θ, 0 and $\frac{1}{2}\,\pi$. Hence the required chance is

$$p = \frac{\int_0^{\frac{1}{2}\pi}\int_0^1 \left\{ \int_{y=0}^{y=1+x\sin\theta} 2(\pi - \phi)d\,s + \int_{y=0}^{y=1-x\sin\theta} 2(\pi - \phi)d\,s \right\} 2\pi x\,d x\,d\theta}{\int_0^{\frac{1}{2}\pi}\int_0^1 \left\{ \int_{y=0}^{y=1+x\sin\theta} 2\pi\,d s + \int_{y=0}^{y=1-x\sin\theta} 2\pi\,d\,s \right\} 2\pi x\,d x\,d\theta}$$

$$=\frac{2}{\pi^3}\int_0^{\frac{1}{2}\pi}\int_0^1\left\{[\pi s-V\psi]_{y=0}^{y=1+x\sin\theta}+[\pi s-V\psi]_{y=0}^{y=1-x\sin\theta}\right\}2x\,dx\,d\theta$$

$$=\frac{4}{\pi^3}\int_0^{\frac{1}{2}\pi}\int_0^1\left\{\pi\sin^{-1}m-[\sin^{-1}m]^2+m^2\right\}2x\,dx\,d\theta$$

$$=\frac{1}{\pi^3}\int_0^{\frac{1}{2}\pi}[\pi^2-4\theta^2+2\theta^2\operatorname{cosec}^2\theta-4\theta\cot\theta+5+\cos 2\theta]\,d\theta$$

$$=\frac{1}{\pi^3}[\pi^2\theta-\frac{4}{3}\theta^3-2\theta^2\cot\theta+5\theta+\tfrac{1}{2}\sin 2\theta]_0^{\frac{1}{2}\pi}=\frac{1}{3}+\frac{5}{2\pi^2}.$$

16. "What are the sides of a right angled triangle the area of which is 5 acres, and one of its acute angles 22° 30′?"

SOLUTION BY E. B. SEITZ, GREENVILLE, OHIO.

Let x = the side adjacent to the given angle; then $x \tan 22° 30'$ = the side opposite, and the area of the triangle is

$$\frac{1}{2}x^2\tan 22° 30' = 800;\ \text{whence}$$

$$x^2 = 1600\cot 22° 30' = 1600\left(\frac{1+\cos 45°}{\sin 45°}\right) = 1600(\sqrt{2}+1);$$

$$\therefore x = 40\sqrt{\sqrt{2}+1},\ \text{and}\ x\tan 22° 30' = 40\sqrt{\sqrt{2}-1}.$$

17. "Find the sum of the series

$$ar^n+(a+b)r^{n-1}+(a+2b)r^{n-2}+\ldots(a+nb)=S".$$

SOLUTION BY PROF. D. TROWBRIDGE, WATERBURGH, N. Y.

Multiply the given series by r and subtract the given series from the product, then

$$S(r-1) = ar^{n+1}-nb+br(r^{n-1}+r^{n-2}+\ldots+1)-a$$

$$\therefore S=\frac{a(r^{n+1}-1)}{r-1}+\frac{br(r^n-1)}{(r-1)^2}-\frac{nb}{r-1}.$$

If $b=0$, we have the ordinary geometrical series.

18. "Find the differential coefficient of the function $(1+x^m)^n.(1+x^n)^m$ and express the answer in its simplest form."

SOLUTION BY PROF. A. W. PHILLIPS, CHESHIRE, CT.

$$\text{Put } u = (1 + x^m)^n . (1 + x^n)^m .$$

$$\frac{du}{dx} = n\,m\,x^{m-1}(1 + x^m)^{n-1}(1 + x^n)^m + n\,m\,x^{n-1}(1 + x^n)^{m-1}(1 + x^m)^n ,$$

$$\frac{du}{dx} = n m u \left\{ \frac{x^{m-1}}{(1+x^m)} + \frac{x^{n-1}}{(1+x)^n} \right\}.$$

19. "Referring to question 4, (No. 1): At what time will the lower cask contain the greatest quantity of wine?"

SOLUTION BY M. BAKER, U. S. C. S. SANFRANCISCO, CAL.

From Mr. S. W. Salmon's solution (No. 3), we have

$$Q = c\,t\,e^{-\frac{c t}{a}}$$

in which Q equals the amount of wine in the lower cask at the end of t minutes. Now the question is what value of t renders Q a maximum?

$$\text{Differentiating we have } \frac{d\,Q}{d\,t} = c\,e^{-\frac{c t}{a}} - \frac{c^2}{a}\,t\,e^{-\frac{c t}{a}}.$$

Making this equal zero and solving, $t = \dfrac{a}{c}$, *i. e.* there will be the greatest amount of wine in the lower cask when all the pipes have been open for the time $\dfrac{a}{c}$ and the quantity of wine in the lower cask at the end of $\dfrac{a}{c}$ minutes is $\dfrac{a}{e}$ ($=$ .36788a).

SOLUTION BY PROF. D. M. SENSENIG, MILLERSVILLE, PA.

Let x represent the number of instants in a minute and u the quantity of wine in the lower cask at the end of t minutes, then will

$$u = t\,c\left(1 - \frac{c}{a\,x}\right)^{t x - 1}, \quad \text{which is a maximum.}$$

$$\frac{d\,u}{d\,t} = \left\{ 1 + t\,x \log' \left(1 - \frac{c}{a\,x}\right) \right\} \left\{ 1 - \frac{c}{a\,x} \right\}^{t x - 1} = 0$$

$$\therefore 1 + t\,x \log' \left(1 - \frac{c}{a\,x}\right) = 0,$$

$$\text{or } 1 - \frac{t\,c}{a} = 0 \text{, since } x = \infty; \quad \text{whence } t = \frac{a}{c}.$$

In solving the above problem, I observed the following beautiful law: Suppose the number of casks containing water to be indefinitely increased,

and the other conditions to remain the same. Develop into a series the expression $\left\{\left(1-\frac{c}{a\,x}\right)+\frac{c}{a\,x}\right\}^{t\,x}$, in which x represent the number of instants in a minute, thus:

$$\left\{\left(1-\frac{c}{a\,x}\right)+\frac{c}{a\,x}\right\}^{t\,x}=\left(1-\frac{c}{a\,x}\right)^{t\,x}+t\,x\left(1-\frac{c}{a\,x}\right)^{t\,x-1}\frac{c}{a\,x}$$
$$+\frac{t\,x(t\,x-1)}{2}\left(1-\frac{c}{a\,x}\right)^{t\,x-2}\frac{c^2}{a^2x^2}+\text{etc.}$$

If we now represent the parts of wine in each cask at the end of t minutes, beginning with the first, respectively by u, u', u'', u''' ets., then will

$u=\left(1-\frac{c}{a\,x}\right)^{t\,x}$, a minimum when $t=\infty$

$u'=t\,x\left(1-\frac{c}{a\,x}\right)^{t\,x-1}\frac{c}{a\,x}$, a maximum when $t=\frac{a}{c}$

$u''=\frac{t\,x(t\,x-1)}{2}\left(1-\frac{c}{a\,x}\right)^{t\,x-2}\frac{c^2}{a^2x^2}$, a maximum when $t=\frac{2\,a}{c}$

$u'''=\frac{t\,x(t\,x-1)(t\,x-2)}{2.\ 3}\left(1-\frac{c}{a\,x}\right)^{t\,x-3}\frac{c^3}{a^3x^3}$ a maximum when $t=\frac{3\,a}{c}$

etc., etc., etc.

The values of u, u' u'', u''', etc., may readily be found by developing the above expressions by the binomial theorem and substituting $x=\infty$ or by applying the principles of logarithms as the student may prefer.

PROBLEMS.

25. By R. L. Seldon, Troy, N. Y.—Required the sides of an obtuse angled triangle the area of which is 14.048 acres, the obtuse angle 111°15′, and one of the acute angles 11° 44′ 10″.

26. By Wm. Hoover, South Bend, Ind.—Find θ from the equation $15\sin\theta+12\cos\theta=17.97240$..................................(1).

27. By D. J. McAdam, Washington, Pa.—Four given equal spheres being placed in close contact with each other, it is required to find the volume of the space inclosed between them and the three triangular planes through each three centers.

29. By Prof. A. B. Evans, Lockport N. Y.—If $a, b, c, d, e, f, g, h, i, j, k$ be chords drawn from any point on the circumference of a circle to the eleven angles of an inscribed regular polygon of eleven sides; prove that $(a+k)(b+j)(c+i)(d+h)(e+g)=f^5$................(1)

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J. E. HENDRICKS, A. M.

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GRAPHICAL SOLUTION OF CUBIC EQUATIONS.

BY A. W. PHILLIPS, CHESHIRE, CT.

Every numerical cubic equation, whose coefficients are real, which is not of the form $x^3 - a\,x + b = 0$ or $x^3 + a\,x + b = 0$, where the intrinsic value of a is positive, but b either positive or negative, may be reduced to one of these forms by known rules.

If a is not zero, the first of the above equations may be transformed into

$$x'^3 - x' + \frac{b\sqrt{a}}{a^2} = 0$$

by substituting for x, $\sqrt{a}.\,x'$, dividing the resulting equation by $a\sqrt{a}$, and afterwards making the denominator of the constant term rational. We put y equal to the left hand member of this last equation and plot the curve when the term $b\sqrt{a} \div a^2$ is zero.

Now, whatever the value of $b\sqrt{a} \div a^2$, the same curve will satisfy the equation provided the origin is removed upon the axis of y a distance equal to the numerical value of this quantity with its sign changed. We draw a line through this new position of the origin parallel to the primitive axis of x'. The real roots of the equation

$$x'^3 - x' + \frac{b\sqrt{a}}{a^2} = 0$$

are the distances on this line intercepted between its intersection with the curve and the new position of the origin, and may be determined by careful measurement with a good degree of accuracy upon a well drawn chart. The values of x may be computed from the equation $x = \sqrt{a}.\,x'$. The equation $x^3 + a\,x + b = 0$ may be transformed into

$$x'^3 + x' + \frac{b\sqrt{a}}{a^2} = 0$$

and its real root found for any value of $b\sqrt{a} \div a^2$ by a process similar to that employed in

$$x'^3 - x' + \frac{b\sqrt{a}}{a^2} = 0.$$

Both the real and imaginary roots of cubic equations may be found with considerable accuracy by means of a table constructed from the equations

$$x'^3 - x' + \frac{b\sqrt{a}}{a^2} = 0 \text{ and } x'^3 + x' + \frac{b\sqrt{a}}{a^2} = 0$$

in tne following manner: In

$$x'^3 - x' + \frac{b\sqrt{a}}{a^2} = 0$$

we can put for $b\sqrt{a} \div a^2$, $\pm c\,(c^2 - 1)$ where $\mp c$ is a root cf the equation. Dividing $x'^3 - x' \pm c\,(c^2 - 1)$ by $x' \pm c$ we obtain $x'^2 \mp c\,x' + (c^2 - 1)$, which put equal to zero and solved as a quadratic gives the other roots. Beginning with some small quantity and assuming in succession values for c differing from each other as little as we please, a table can be computed showing the values of $b\sqrt{a} \div a^2$ and the three corresponding roots to any degree of accuracy.

When $b\sqrt{a} \div a^2 = \pm .385+$ we have the case of two equal roots, and for greater values of $b\sqrt{a} \div a^2$ there is but one real root, and the other two are imaginary.

Designating the pair of imaginaries corresponding to the real root $+c$ by $-A \pm B\sqrt{(-1)}$, we may compute the values of A and B for the table. The imaginary roots answering to the various values of $b\sqrt{a} \div a^2$ may be represented on the chart. By the principles of the General Theory of Equations the sum of the three roots of the equation

$$x'^3 - x' + \frac{b\sqrt{a}}{a^2} = 0$$

is equal to zero; hence the A of the imaginary is equal to one-half the real root with the opposite sign, and may be laid off on the same line as the real root but on the opposite side of the axis of y. If, at the extremity of this line, a perpendicular be drawn and B be laid off upon this perpendicular in both directions, we have a complete representation of the imaginary roots. By laying off a sufficient number of these values, we are able to draw the curve which represents the imaginary roots.

It will he observed that as the two roots on the left of the axis of y

approach equality their values change rapidly in comparison with the value of $b\sqrt{a} \div a^2$. The same holds true for the value of B in the case of the first imaginaries. But as $b\sqrt{a} \div a^2$ increases, the changes in the real and imaginary roots grow relatively smaller, and results of great accuracy may be obtained by interpolation.

In the equation

$$x'^3 + x' + \frac{b\sqrt{a}}{a^2} = 0$$

the computation of $b\sqrt{a} \div a^2$ for the table may be abridged, by taking into account the fact that the third order of differences in the series of terms $b\sqrt{a} \div a^2$ corresponding to the values of $c = .1, .2, .3$ *&c.* vanishes, and B of the imaginaries varies very uniformly.

To use the table for numerical computation, reduce the eqeation to one of the two forms

$$x'^3 - x' + \frac{b\sqrt{a}}{a^2} = 0 \text{ or } x'^3 + x' + \frac{b\sqrt{a}}{a^2} = 0,$$

compute $b\sqrt{a} \div a^2$ by means of a table of squares and square roots, find this number in the proper table in the column

$$\text{“}\frac{b\sqrt{a}}{a^2}\text{”},$$

and opposite will be found the roots, if real. If there is a pair of imaginaries the columns A and B will furnish their values. If $b\sqrt{a} \div a^2$ is not found in the table, find the next less number, and add to the corresponding roots proportional parts.

The following is a portion of a table constructed as described above :

TABLE.

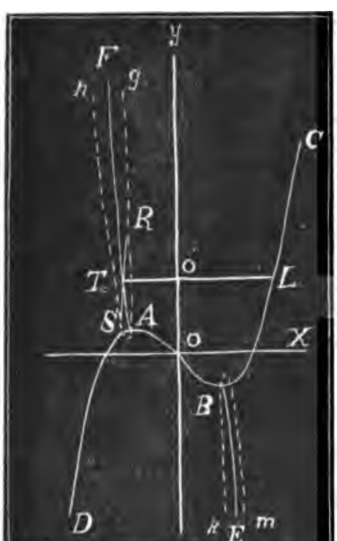
$$x'^3 - x' + \frac{b\sqrt{a}}{a^2} = 0.$$

$\pm\frac{b\sqrt{a}}{a^2}$	REAL ROOTS.			$\pm\frac{b\sqrt{a}}{a^2}$	$A.$	$B.$	REAL ROOTS.
±.000	±.000	±1.000	∓1.000	± .386+	±.5775	.024+	∓1.155
.099	.100	.946+	1,046+	.401+	.580	.096+	1.160
.192	.200	.885+	1.085+	.463+	.590	.215+	1.180
.273	.300	.816+	1.116+	.528+	.600	.283+	1.200
.336	.400	.738+	1.138+	.595	.620	.341+	1.220
.359	.450	.696+	1.146+	.817	.640	.479+	1.280
.375	.500	.651+	1.151+	.980	.660	.554+	1.320
.377+	.510	.642+	1.152+	1.155	.680	.622+	1.360
.379+	.520	.633+	1.153+	1.344	.700	.685+	1,400
.381+	.530	.624+	1.153+	2.496	.800	.959+	1.600
.383+	.540	.614+	1.154+	4.032	.900	1.196+	1.800
.384+	.550	.602+	1.154+	6.000	1.000	1.414+	2.000
.385+	.577+	.577+	1.154+	24.000	1.500	2.398+	3.000

$$x'^3 + x' + \frac{b\sqrt{a}}{a^2} = 0.$$

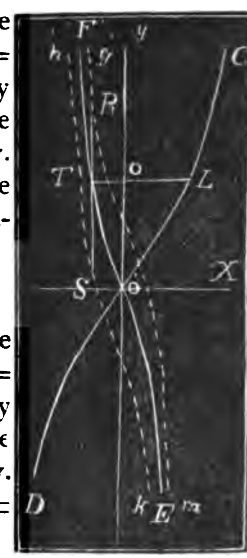
$\pm\frac{b\sqrt{a}}{a^2}$	$A.$	$B.$	REAL ROOT.	$\pm\frac{b\sqrt{a}}{a^2}$	$A.$	$B.$	REAL ROOT.
.000	±.000	±1.000	∓.000	2.000	±.500	±1.323+	∓1.000
.101	.050	1.004+	.100	2.431	.550	1.381+	1.100
.208	.100	1.015+	.200	2.928	.600	1.442+	1.200
.327	.150	1.033+	.300	3.497	.650	1.506+	1.300
.464	.200	1.058+	.400	4.144	.700	1.572+	1.400
.625	.250	1.090+	.500	4.875	.750	1.640+	1.500
.816	.300	1.128+	.600	5.696	.800	1.709+	1.600
1.043	.350	1.170+	.700	6.613	.850	1.780+	1.700
1.312	.400	1.216+	.800	7.632	.900	1.852+	1.800
1.629	.450	1.268+	.900	10.000	1.000	2.000	2.000

$$x'^3 - x' + \frac{b\sqrt{a}}{a^2} = 0 \qquad\qquad x'^3 + x' + \frac{b\sqrt{a}}{a^2} = 0$$



$C\ B\ A\ D$ = curve whose equation is $y = x'^3 - x'$. $O'\ L$ = any real root. $O'\ T$ = the "A" of an imaginary. $T\ R$ and $T\ S$ = the "$\pm\ B$" of an imaginary.

$C'\ O\ D'$ = curve whose equation is $y = x'^3 + x'$. $O'\ L$ = any real root. $O'\ T$ = the "A" of an imaginary. TR and TS = the "$\pm B$" of an imaginary.



Example. $x^3 - 3\,x - 3 = 0$.

Put $x = \sqrt{3}.\ x'$ and substitute, $3\sqrt{3}.\,x'^3 - 3\sqrt{3}.\,x' - 3 = 0$. Dividing by $3\sqrt{3}$ and reducing, gives $x'^3 - x' - .577 + = 0$, where $- .577 + = b\sqrt{a} \div a^2$. The real root in the table answering to the next less value of $b\sqrt{a} \div a^2$ .528 is 1.20. By proportional parts $x' = 1.214 +$. Proceeding in a similar way the imaginary roots may be found, and by substituting the values of x' in the equation $x = \sqrt{3}.\,x'$, the values of x may be obtained.

ON THE EXTRACTION OF ROOTS OF WHOLE NUMBERS BY THE "BINOMIAL THEOREM."

BY ARTEMAS MARTIN, MATHEMATICAL EDITOR SCHOOLDAY MAGAZINE, ERIE, PA.

In many instances the "Binomial Theorem" may be employed with advantage in the extraction of roots of whole numbers.

I.—We have

$$(a^n+x)^{\frac{1}{n}}=a+\frac{x}{n\,a^{n-1}}-\frac{(n-1)x^2}{1.\,2.\,n^2.\,a^{2n-1}}+\frac{(n-1)(2n-1)x^3}{1.\,2.\,3.\,n^3.\,a^{3n-1}}$$

$$-\frac{(n-1)(2n-1)(3n-1)x^4}{1.\,2.\,3.\,4.\,n^4.\,a^{4n-1}}+\&c.\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

and

$$(a^n-x)^{\frac{1}{n}}=a-\frac{x}{n\,a^{n-1}}-\frac{(n-1)x^2}{1.\,2.\,n^2.\,a^{2n-1}}-\frac{(n-1)(2n-1)x^3}{1.\,2.\,3.\,n^3.\,a^{3n-1}}$$

$$-\frac{(n-1)(2n-1)(3n-1)\,x^4}{1.\,2.\,3.\,4.\,n^4.\,a^{4n-1}}-\&c.\ldots\ldots\ldots\ldots(2).$$

These formulas are very convenient when x is small compared with a.

EXAMPLES.—1. Required the square root of 26. Making $a=5$, $x=1$ and $n=2$, in (1),

$$\sqrt{26}=5+\frac{1}{10}-\frac{1}{1000}+\frac{1}{50000}-\frac{1}{2000000}+\&c.,$$

$$=5.099019+.$$

2.—Required the square root of 216.

$$216=24\times 9;\quad\therefore\sqrt{216}=3\sqrt{24}.$$

By (2),

$$\sqrt{24}=5-\frac{1}{10}-\frac{1}{1000}-\frac{1}{50000}-\frac{1}{2000000}-\&c.,$$

$$=4.8989794+;$$

$$\therefore\ \sqrt{216}=14.696938+.$$

3.—Required the fifth root of 30.

$$\sqrt[5]{30}=\sqrt[5]{32-2}=2-\frac{1}{5.2^3}-\frac{1}{5^2.2^6}-\frac{3}{5^3.2^{10}}-\&c.,$$

$$=1.974351+.$$

Also,

$$(a^n + x)^{-\frac{1}{n}} = \frac{1}{a} - \frac{x}{n\,a^{n+1}} + \frac{(n+1)\,x^2}{1.\,2.\,n^2.\,a^{2n+1}} - \frac{(n+1)\,(2n+1)x^3}{1.\,2.\,3.\,n^3.\,a^{3n+1}}$$

$$+ \frac{(n+1)\,(2\,n+1)\,(3\,n+1)x^4}{1.\,2.\,3.\,4.\,n^4.\,a^{4n+1}} - \&\text{c}\ldots\ldots(3),$$

and

$$(a^n - x)^{-\frac{1}{n}} = \frac{1}{a} + \frac{x}{n\,a^{n+1}} + \frac{(n+1)\,x^2}{1.\,2.\,n^2.\,a^{2n+1}} + \frac{(n+1)\,\,(2\,n+1)x^3}{1.\,2.\,3.\,n^3.\,a^{3n+1}}$$

$$+ \frac{(n+1)\,(2\,n+1)\,(3\,n+1)x^4}{1.\,2.\,3.\,4.\,n^4.\,a^{4n+1}} + \&\text{c}\ldots\ldots(4).$$

Example.—Required the one-hundredth root of 2.

$$2 = \frac{1}{1-\frac{1}{2}} = (1-\tfrac{1}{2})^{-1},\ \therefore (2)^{\frac{1}{100}} = (1-\tfrac{1}{2})^{-\frac{1}{100}}.$$

Therefore by (4)

$$(2)^{\frac{1}{100}} = 1 + \frac{1}{200} + \frac{101}{80000} + \frac{6767}{16000000} + \frac{2036867}{12800000000} + \&\text{c.},$$

$$= 1.00695 +,\text{ using 9 terms.}$$

II.—Let a be any number, then

$$a = \frac{a\,p^n}{p^n} \times \frac{q^n}{q^n} = \frac{p^n}{q^n}\left[1 - \left(\frac{p^n - a\,q^n}{p^n}\right)\right];$$

$$\therefore (a)^{\frac{1}{n}} = \frac{p}{q}\left[1 - \left(\frac{p^n - a\,q^n}{p^n}\right)\right]^{\frac{1}{n}},$$

$$= \frac{p}{q}\left[1 - \left(\frac{p^n - a\,q^n}{np^n}\right) - \frac{(n-1)}{1.\,2}\left(\frac{p^n - a\,q^n}{n\,p^n}\right)^2\right.$$

$$\left. - \frac{(n-1)\,(2\,n-1)}{1.\,2.\,3}\left(\frac{p^n - a\,q^n}{n\,p^n}\right)^3 - \frac{(n-1)(2n-1)(3n-1)}{1.\,2.\,3.\,4}\left(\frac{p^n - a\,q^n}{n\,p^n}\right)^4 - \&\text{c.}\right],$$

where p and q may be any numbers chosen at pleasure.

Let r_m be the root of a to m places of decimals, and R_m the remainder; then if $p = (10)^m$ and

$$q = \frac{(10)^m\ r_m}{a},$$

$$(a)^{\frac{1}{n}} = \frac{(10)^m}{(10)^m\left(\frac{r_m}{a}\right)}\left[1 - \frac{\left(\frac{R_m}{a}\right)}{(10)^{2m}}\right]^{\frac{1}{n}}.$$

Take $a = 2$, $n = 2$ and $m = 8$; then

$$\sqrt{2} = \frac{100000000}{70710678}\left[1 - \frac{33560632}{10000000000000000}\right]^{\frac{1}{2}},$$

$$= \frac{100000000}{70710678}\left[1 - \frac{16780316}{10000000000000000}\right.$$

$$- \frac{140789502529928}{(10000000000000000)^2} - \frac{236249234193499129724 8}{(10000000000000000)^3}$$

$$- \frac{4955421005656150678133921296}{(10000000000000000)^4}$$

$$\left. - \frac{1164149425431271940117820855324133504}{(10000000000000000)^5} - \&c.\right],$$

$$= 1.4142135623730950488016887242096980785696718753769 4 +.$$

Let $a = 3$, $n = 2$ and $m = 13$; then

$$\sqrt{3} = \frac{10000000000000}{5773502691896}\left[1 - \frac{8925087775552}{100000000000000000000000000}\right]^{\frac{1}{2}}$$

$$= \frac{10000000000000}{5773502691896}\left[1 - \frac{4462543887776}{100000000000000000000000000}\right.$$

$$- \frac{9957148975163468441113088}{(100000000000000000000000000)^2}$$

$$\left. - \frac{44434214298790798522333481840396812288}{(100000000000000000000000000)^3} - \&c.\right],$$

$$= 1.7320508075688772935274463415058723669428052538103 8 +.$$

Four additional terms would give the root true to at least one hundred places of decimals.

ON THE STABILITY OF THE SOLAR SYSTEM.

BY R. J. ADCOCK, MONMOUTH, ILL.

The resultant of parallel forces passes through the centre of inertia. The forces of attraction of the particles of one body for another, not being generally parallel, their resultant does not pass through their centres of inertia, except in case of spheres and bodies whose particles are symmetrically disposed with refference to the line joining their centres of inertia.

Hence, with these exceptions, the attraction between two bodies has a tendency to give each a rotary motion about its centre of inertia.

The resultant of the forces of attraction of the moon for the earth, in consequence of the tides raised by the moon and the fact that the highest tide is always to the east of the moon, passes about 23 inches to the east of the earth's centre, thereby continually diminishing the rotary motion of the earth on its axis, lengthening the day and diminishing the year as measured thereby, as accurate astronomical observations and calculations must show. The length of the year as determined in the time of J. Cæsar, about 45 B. C., was 365¼ days, which at present is estimated at between 11 and 12 minutes less.

In like manner as the attraction of the moon upon the figure of the earth and the position of that figure as caused by the tides which that attraction produces, continually diminish the rotary motion of the earth on its axis, so the attraction of the sun, must produce a like effect upon the rotary motion of the earth and moon about their common centre of inertia.

When a body revolves about a horizontal axis, to which it is attached by means of an inextensible string, its motion, when acted upon only by the force of gravity, is permanent; in case the string is either partially or perfeetly elastic, or the body of such elastic material that it is lengthened by the greater velocity in the lower part of its path, its motion will not be permanent, even when there is no resistance from friction or the atmosphere. In consequence of the greater extension being always after it passes the point directly under the axis, in ascending the retardation by the force of gravity is always greater than the accelleration in descending on the other side.

This fact, of the change in the direction and amount of the force of attraction caused by the change of torm in consequence of fluieity, is

not taken into the account in the article by Mr. Hill, neither is it, in the reasonings of Laplace on the time of the rotary motion of the earth on its axis, nor in the discussion of the subject of perturbations.

The bearing of fluidity upon the stability of the system and the great changes which are to take place, some probably in consequence thereof, are very clearly referred to in the book of Psalms, in the Prophets, by our Saviour himself in the Gospels of Matthew and Mark, by the Apostle Peter, and in the book of Revelation of the Apostle John.

ON THE SUMMATION OF SERIES.

BY EDGAR FRISBY, ESQ., U. S. NAVAL OBSERVATORY, WASHINGTON, D. C.

Suppose we have a series whose n^{th} term is of the form

$$\frac{u_m}{(ax+b)(ax+a+b)(ax+2a+b)\ldots(ax+\overline{n-1}a+b)}$$

where u_m is an algebraic function of x of not greater than $n-2$ dimensions: it can be summed very simply by an ordinary algebraic method in this manner:

Resolve the general term into partial fractions of the form

$$\frac{A}{ax+b}+\frac{B}{ax+a+b}+\frac{C}{ax+2a+b}+\ldots\text{\&c.},$$

then we shall have $Au'_{n-1}+Bu''_{n-1}+Cu'''_{n-1}+\ldots\text{\&c}=u_m$ where $u'_{n-1}, u''_{n-1}, u'''_{n-1}$ &c. are algebraic functions of x of the $n-1^{th}$ degree. If now these expressions be expanded, and we equate the coefficients of x^{n-1} we shall have in the case supposed $\because$ m is less than $n-1$

$a^{n-1}(A+B+C+\text{\&c.})=0$ or $A+B+C+\text{\&c}=0$.

$\therefore$ for $x=1, 2, 3, 4$ &c. we shall have

$$\begin{aligned}
&A\left\{\frac{1}{a+b}+\frac{1}{2a+b}+\frac{1}{3a+b}+\frac{1}{4a+b}+\frac{1}{5a+b}+\text{\&c.,}\right\}\\
&+B\left\{\qquad\qquad\frac{1}{2a+b}+\frac{1}{3a+b}+\frac{1}{4a+b}+\frac{1}{5a+b}+\text{\&c.,}\right\}\\
&+C\left\{\qquad\qquad\qquad\qquad\frac{1}{3a+b}+\frac{1}{4a+b}+\frac{1}{4a+b}+\text{\&c.,}\right\}
\end{aligned}$$

whose summation will consist of a few only of the first and last terms.

As an example, let $n=5$, $m=3$, $a=1$ & $b=8$ and take the series

$$\frac{9}{9.10.11.12.13}+\frac{22}{10.11.12.13.14}+\frac{47}{11.12.13.14.15}+\frac{90}{12.13.14.15.16}+$$
&c., and let it be required to sum it to n terms and to infinity. We first find the scale of relation of the numerator to be $x^3 + 6x + 2$, which satisfies the condition of the problem. Let

$$\frac{A}{x+8}+\frac{B}{x+9}+\frac{C}{x+10}+\frac{D}{x+11}+\frac{E}{x+12}=$$
$$\frac{x^3+6x+2}{(x+8)(x+9)(x+10)(x+11)(x+12)}$$

$A(x+9)(x+10)(x+11)(x+12)+B(x+8)(x+10)(x+11)$ $(x+12)+\ldots\text{\&c.}=x^3+6x+2.$

By making $x=-8,-9,-10,-11,-12$, respectively we have
$A=-\frac{279}{12}$ $B=+\frac{1562}{12}$ $C=-\frac{3174}{12}$ $D=+\frac{2790}{12}$ $E=-\frac{899}{12}$

and the series becomes, if S represents the sum,

$$\begin{aligned}12S=&-279\left(\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\frac{1}{14}+\ldots+\frac{1}{n+8}\right)\\
&+1562\left(\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\frac{1}{14}+\ldots\frac{1}{n+8}+\frac{1}{n+9}\right)\\
&-3174\left(\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\frac{1}{14}\ldots\frac{1}{n+8}+\frac{1}{n+9}+\frac{1}{n+10}\right)\\
&+2790\left(\frac{1}{12}+\frac{1}{13}+\frac{1}{14}\ldots\frac{1}{n+8}+\frac{1}{n+9}+\frac{1}{n+10}+\frac{1}{n+11}\right)\\
&-899\left(\frac{1}{13}+\frac{1}{14}+\ldots\frac{1}{n+8}+\frac{1}{n+9}+\frac{1}{n+10}+\frac{1}{n+12}\right.\\
&\left.+\frac{1}{n+12}\right)\end{aligned}$$

$$\begin{aligned}12S=&\left(\frac{-279}{9}+\frac{279}{n+9}\right)+\left(\frac{1283}{10}-\frac{1283}{n+10}\right)\\
&-\left(\frac{1891}{11}-\frac{1891}{n+11}\right)+\left(\frac{899}{12}-\frac{899}{n+12}\right)\end{aligned}$$

to n terms,

$$\begin{aligned}12\Sigma=&-\frac{279}{9}+\frac{1283}{10}-\frac{1891}{11}+\frac{899}{12}=-31+128-172+83\\
&+\frac{3}{10}+\frac{1}{11}+\frac{1}{4}=8\tfrac{141}{220};\end{aligned}$$

for an infinite No. of terms

$$\therefore\ \Sigma=\frac{8\frac{141}{220}}{12}=\frac{1901}{2640}.$$

ANSWER TO QUERY IN NO. 3.

BY PROF. E. W. HYDE, CHESTER, PA.

QUERY.—"The strain on the internal surface of a steam boiler tending to produce longitudinal rupture, is, per unit of length, equal to the product of the elastic pressure of the steam per unit of surface multiplied by the diameter (?) of the boiler. What should be the relative thickness of the flat ends of a boiler, so that, without bracing, they are just strong enough to withstand the maximum strain that the boiler will bear."

In the first place the word *diameter* used above should be *radius.**

Let p = pressure per unit of surface.

h = thickness of end plate.

f = modulus of rupture for breaking across.

f_1 = " " " " longitudinal tension.

The formula for the strength of a rectangular beam uniformly loaded is $P = p\,b\,l = \frac{4}{3}.\frac{f\,b\,h^2}{l}$.

We may consider the portion of the end plate whose length is $2\,x$, breadth $d\,y$, and thickness h, as such a beam. The pressure on this elementary beam will be $= 2\,p\,x\,d\,y$.

$$\therefore \quad 2\,p \int_{o}^{r} x\,d\,y = \frac{4}{3} f\,h^2 \int_{o}^{r} \frac{d\,y}{2\,x}, \quad \text{or}$$

*Because the strength of the metal is supposed to be uniform, we may assume that rupture occurs simultaneously along two opposite sides of the boiler, (for when the force is sufficient to produce rupture at any one point, it is obviously just sufficient to produce rupture at the opposite point).

Imagine a plane, therefore, to separate the cavity of the boiler longitudinally into two equal parts; we may assume that rupture takes place along the lines that limit the two edges of this imaginary plane; and the force which produces these two lines of rupture is obviously measured by the total reaction of the steam in either half of the boiler, on the above named imaginary plane. Hence, though it is true that to produce a single line of rupture would require a force equivalent to the elastic pressure of the steam per unit of surface multiplied by the *radius* of the boiler, yet it is equally true that the force which *produces* rupture is represented by the total reaction of the steam in either half of the boiler on the above named imaginary plane, and is therefore "equal to the product of the elastic pressure of the steam, per unit of surface, multiplied by the *diameter* of the boiler." Practically, however, it is immaterial whether we conceive the boiler to be ruptured at two opposite points as a result of the total pressure, or whether we regard only a single line of rupture as resulting from half the pressure.—ED.

$$3p\int_0^r x\,dy = fh^2\int_0^r \frac{dy}{x}.$$

But from the equation of the circle we have

$$x = \sqrt{r^2 - y^2}.$$

$$\therefore \quad 3p\int_0^r dy\sqrt{r^2-y^2} = fh^2\int_0^r \frac{dy}{\sqrt{r^2-y^2}}.$$

Whence by integration,

$$3p\,.\frac{\pi r^2}{4} = fh^2.\frac{\pi}{2}.$$

$$\therefore \quad h = r\sqrt{\frac{3p}{2f}}.$$

The formula for the thickness of the cylindrical portion of the boiler is, if $t =$ the thickness,

$$t = \frac{pr}{f_1}.$$

Comparing these two expressions we see that

$$\frac{h}{t} = r\sqrt{\frac{3p}{2f}} \div \frac{pr}{f_1} = \frac{rf_1}{rp}\sqrt{\frac{3p}{2f}} = f_1\sqrt{\frac{3}{2pf}}.$$

This formula gives the thickness h much greater than t, as must evidently be the case, though since p is in the denominator this difference will diminish as p increases..

[We have inserted the foregoing answer to "Query" in No. 3 because it is the only answer that has been received. We think however that the investigation is defective, because it is virtually assumed that the sum of the capacities of the elementary beams to support a load uniformly distributed on each, divided by the area of the end of the boiler, is its capacity per unit of surface. This assumption has not been proved and is probably not true. For, though the lateral connection of the elementary beams serves to equalize their capacity to some extent, yet it is not likely that it exactly proportions the capacity of each elementary beam to its length, as it must if the result obtained be rigorously true. On the contrary, it is highly improbable that if equally pressed over the whole of its internal surface the end of a boiler would be equally liable to yield at every point of its surface. The subject is one of practical importance, however, and as we are not aware that it has been fully discussed heretofore, we would be pleased to have it thoroughly investigated in our pages.]

SOLUTIONS OF PROBLEMS IN NO. 5.

Solutions of problems in No. 5 have been received as follows: From Prof. P. E. Chase, 24; S. J. Child, 21 & 22; Geo. M. Day, 20, 21 & 22; Prof. A. B. Evans, 20, 21, 22, 23 & 24; Henry Gunder, 20, 21, 22 & 23; Prof. E. W. Hyde, 20; H. Heaton, 24; Artemas Martin, 20, 21, 22 & 23; Miss Esther Matthews, 21, 22 & 23; L. E. Newcomb, 20, 21, 22 & 23; A. W. Phillips, 20, 21, 22 & 23; Henry A. Roland, 20, 21, 22, 23 & 24; L. Regan, 21 & 22; S. W. Salmon, 20, 21 & 23; Walter Sirerly, 20, 21, 22 & 23; T. P. Stowell, 20 & 22; Prof. D. M. Sensenig, 20 & 21; E. B. Seitz, 20, 21, 22 & 23; Prof. J. Scheffer, 20, 21, 22 & 23; Prof. H. T. J. Ludwick, 21 & 22.

We received elegant solutions of all the problems in No. 4, from Artemas Martin, but by an oversight failed to give him credit for them in No. 6.

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20. "It is required to circumscribe about a given parabola an isosceles triangle whose area shall be a minimum."

SOLUTION BY ARTEMAS MARTIN, ERIE, PA.

Let a be the altitude of the parabola, and x and y the coordinates of the point of contact with the triangle; then $a + x$ is the altitude of the triangle, and we have by similar triangles

$$2x : 2y :: a + x : \frac{y(a+x)}{x},$$

the base of the triangle.

Its area $= \dfrac{y(a+x)^2}{2x} = \dfrac{(a+x)^2\sqrt{2px}}{2x}$, since $y^2 = 2px$.

$$\therefore \frac{(a+x)^2\sqrt{2px}}{2x} = \text{minimum},$$

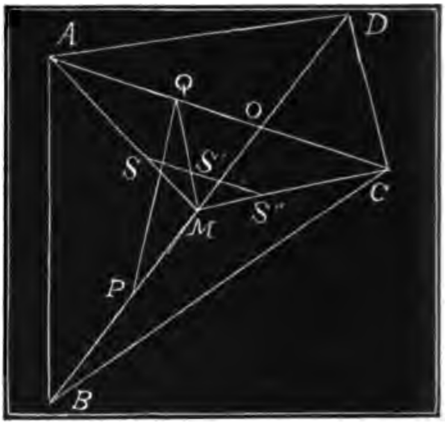
$$\text{or } \frac{(a+x)^4}{x} = \text{minimum} = u.$$

$$\frac{du}{dx} = \frac{4x(a+x)^3 - (a+x)^4}{x^2} = 0.$$

$$\therefore a + x = 4x = \frac{4}{3}a, \text{ the altitude of the triangle.}$$

21. "$A\,B\,C\,D$ is a quadrilateral; O, the intersection of the diagonals; P, $\mathcal{Q}$, points in BD, AC, such that $\mathcal{Q}A = O\,C$ and $P\,B = O\,D$. Prove that the centre of gravity of the quadrilateral coincides with that of the triangle $O\,P\,\mathcal{Q}$."

SOLUTION BY GEO. M. DAY, LOCKPORT, N. Y.



Let M be the middle point of $D\,B =$ middle of $P\ O$. Take $M\,S = \frac{1}{3}\,M\ A$ and draw $S\ S''$ parallel to $A\ C$; join $M\ \mathcal{Q}$, then S, S', S'' are the centres of gravity of $A\,B\,D$, $O\,P\,\mathcal{Q}$, $D\,B\,C$ respectively. We have $A\,B\,D : B\,D\,C :: A\,O : O\,C$, and $S'\,S'' : S\,S' :: A\,O : O\,C$. $\therefore A\,B\,D : B\,D\,C :: S'\,S'' : S\,S'$, hence S' is the centre of gravity of the quadrilateral $A\,B\,C\,D$.

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22. "Show that the distance from a vertex of any plane triangle to the point where the opposite escribed circle touches the sides meeting at the vertex is constant and equal to half the sum of the sides of the triangle."

SOLUTION BY E. B. SEITZ, GREENVILLE, O.

Let $A\,B\,C$ be any triangle, D, E the points at which the escribed circle opposite A touches $A\,B$, $A\ C$ produced. Now $B\,D + C\,E = B\,C$; $\therefore\ A\,D + A\,E = A\,B + A\ C + B\ C$. But $A\,D = A\,E$; $\therefore A\,D = \frac{1}{2}\,(A\,B + A\ C + B\ C)$.

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23. "If the brightness of the moon be equal to the brightness of the clouds by day, show that the light of an overcast day is to that of a full moon-lit night as $8\,(360)^2 : \pi^2$; the diameter of the moon being $30'$."

SOLUTION BY PROF. W. C. ESTY, AMHERST, MASS.

Let $r =$ the radius of the cloud surface.

Then $2\,\pi\,r^2 =$ the visible cloud area,

$\frac{\pi r}{360}$ = the apparent diameter of moon,

$\frac{\pi}{4}\left(\frac{\pi^2 r^2}{360^2}\right) = \frac{\pi^3 r^2}{4.(360)^2}$ = the apparent area of the moon,

or the portion of the cloud surface covered by its disk.

The ratio required is

$$2 \pi r^2 \ : \frac{\pi^3 r^2}{4.(360)^2} = 8.\ (360)^2 : \pi^2.$$

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24. "There are m labels, to be distributed by lot among m different articles. Required the probable amount of coincidence in two independent allotments."

SOLUTION BY PROF. PLINY EARLE CHASE, HAVERFORD COLLEGE, PA.

Suppose the articles to be similarly arranged in the two allotments. Then, among the possible permutations of the labels, there is only one in which they will all coincide, and none in which $m - 1$ will coincide.

If all but two coincide, those two must change places. There are, $\therefore$ as many such arrangements as we can make selections of 2 out of m, or $\frac{m\,(m-1)}{2\,!}$.

If all but three coincide, those three must change places in such a way that neither will occupy its original place. This can be done in two ways with each group of 3; there are, $\therefore$, twice as many such arrangements as the number of selections that we can make of 3 out of m, or $2 \times \frac{m\,(m-1)\,(m-2)}{3\,!}$.

Tabulating and differencing those results, it will be seen that the number of arrangements in which there are n displacements out of m, is

$$\Delta^n\, 0\,! \frac{\times\, m \ldots (m-n+1)}{n\,!}.$$

The value of this expression can be easily found, for

$$\Delta^n\, 0\,! = \frac{n\,!}{0\,!} - \frac{n\,!}{1\,!} + \frac{n\,!}{2\,!} - \ldots$$

$$n\,\Delta^{n-1}\, 0\,! = \frac{n\,!}{0\,!} - \frac{n\,!}{1\,!} + \frac{n\,!}{2\,!} - \ldots (-)^{n+1}\, 1.$$ Hence, by subtraction,

$$\Delta^n\, 0\,! = n\,\Delta^{n-1}\ 0\,!\,(-)^n\, 1$$

There will $\therefore$ be m displacements, or 0 coincidences, in

$$\Delta^m 0 ! = \frac{m !^*}{e} = .36788\, m !$$

arrangements, leaving for the probability of one or more coincidences, .63212 m ! out of m ! possible arrangements,—which is a probability of about $\frac{5}{8}$.

There will also be $m - 1$ displacements, or 1 coincidence, in .36788 m ! arrangements. Deducting this amount from .63212 m ! there remains only a chance of $\frac{26474}{100000}$ for more than one coincidence. There would $\therefore$ probably be one accidental coincidence, and no more.

SOLUTION BY PROF. W. C. ESTY, AMHERST, MASS.

There are $|\underline{m}$ ways in which the labels may be arranged. If we take any particular set of labels, r in number, and designate by A_r the number of arrangements in which this set of labels all fail to coincide and then multiply this number A_r by the number of sets consisting of r labels each that may be formed out of m labels, we shall have the entire number of arrangements in which there are r labels that fail of coincidence. This product

$$= A_r \frac{|\underline{m}}{|\underline{r}\ |\underline{m-r}}.$$

It is evident that the same product gives the number of cases in which there are $m - r$ coincidences. Putting, in the above formula, for r every number in succession from $r = 0$, up to $r = m$ inclusive and taking the sum of the results we have all the arrangements possible $|\underline{m}$ in number *i. e.*

$$|\underline{m} = \sum_{0}^{m}{}_r\, A_r \frac{|\underline{m}}{|\underline{r}\ |\underline{m-r}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

This is the equation given on page 336 of Todhunter's History of the Theory of Probabilities where a general formula for computing A_r is given, which is not needed for our present problem.

The probability that r labels will fail and that $m - r$ labels will not fail to coincide may be designated by P_r which can be found by the equation

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*The student will understand that the notation m ! means the same as $|\underline{m}$, and that each represents 1. 2. 3. . . . . m.—ED.

$$P_r = A_r \frac{\lfloor m}{\lfloor r \ \lfloor m-r} \dots\dots\dots\dots\dots\dots\dots\dots (2)$$

when A_r is known.

Let E_r = the mathematical expectation of coincidence, or the *probability* of an arrangement containing any assigned number of coincidences as $m-r$, multiplied by the number of coincidences. Then will

$$E_r = (m-r)\ P_r \dots\dots\dots\dots\dots\dots\dots\dots (3).$$

Now the total expectation of coincidence will be a summation of all the terms E_r from $r=0$ to $r=m-1$ inclusive. Call this total expectation, or probable amount of coincidence, C. Then

$$C = \sum_{0}^{m-1}{}_r E_r = \sum_{0}^{m-1}{}_r (m-r)\ P_r = \sum_{0}^{m-1}{}_r \frac{(m-r)\ A_r}{\lfloor r \ \lfloor m-r}‡$$

$$= \sum_{0}^{m-1}{}_r \frac{A_r}{\lfloor r \ \lfloor m-1-r} = \frac{1}{\lfloor m-1} \sum_{0}^{m-1}{}_r \frac{A_r \lfloor m-1}{\lfloor r \ \lfloor m-1-r} = \frac{\lfloor m-1}{\lfloor m-1} \text{ [by (1)]}$$

$$=1.$$

PROBLEMS.

30. By Otis Shepard, Goodland, Ind.—Prove that

$$\frac{\sqrt[4]{a}+\sqrt[4]{b}}{\sqrt[4]{a}-\sqrt[4]{b}} = \frac{a+b+2\sqrt{ab}+2\sqrt[4]{a^3b}+2\sqrt[4]{ab^3}}{a-b}.$$

31. By Prof. D. Kirkwood, Bloomington, Ind.—Solve the equation

$$x = \sqrt[4]{x + x\sqrt[4]{x + x\sqrt[4]{x + \&c}}}$$

and express the value of x in a finite number of terms.

32. By Prof. D. Trowbridge, Waterburgh, N. Y.—There are two spheres of equal size and of exactly the same appearance, one solid

---

‡In order that we may obtain the probable number of coincidences *in each independent allotment* the sum, C, must be divided by the number of permutations, $\lfloor m$; $\therefore$

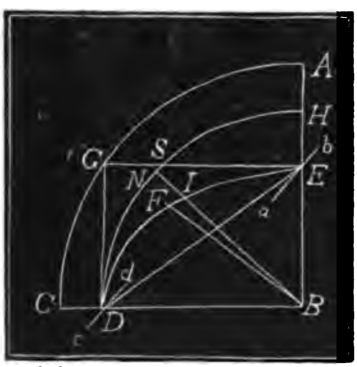
$$\frac{C}{\lfloor m} = \frac{1}{\lfloor m} \sum_{0}^{m-1}{}_r E_r = \frac{1}{\lfloor m} \sum_{0}^{m-1}{}_r (m-r)\ P_r = \sum_{0}^{m-1}{}_r \frac{(m-r)\ A_r}{\lfloor r \ \lfloor m-r}.—\text{Ed.}$$

silver and galvanized with gold, the other hollow and made of gold.
Required some means by which one may be determined from the other.

33. By Prof. W. W. Johnson, St. John's College, Annapolis, Md.—A circle is referred to rectangular axes passing through its centre. A tangent and ordinate are drawn from any point of the circumference. A distance n times the abscissa of this point is measured upon the axis of x, and from this point a perpendicular falls upon the tangent. Required the equation of the locus of the intersection of the perpendicular and tangent.

SOLUTION OF A QUESTION IN MECHANICS.

BY DR. J. B. HOLCOMB, NEWPORT, N. Y.



$A\,B$ and $C\,B$ are two equal lines perpendicular to each other, and $D\,E$ equal to either is inserted between them. Two equal and perfectly elastic balls fall from A and C towards a force at B, which force varies as the distance from B. At D and E the balls are deflected by planes $c\ d$ and $a\ b$ inclined at an angle of 45° to $A\,B$ and $C\,B$.

What will become of the balls?

Will they ever meet, and if so where? After such meeting, if it should happen, what will become of them then?

SOLUTION.

There may be two cases:

1. When the balls are deflected at E and D at the same instant.
2. When they begin their motion at A and C at the same instant.

For the 1st describe on $E\,B$ and $B\,D$ as semi-axes the quadrant of an ellipse $D\,F\,E$. Bisect the area $D\,F\,E\,B$ by the line $B\,F$, then the two balls will meet in the elliptic arc at the point F.

For the line $D\,E$ which joins the extremities of the two axes of the ellipse, being equal to either of the lines $A\,B$ and $C\,B$ is a measure of the force which either of the balls would have acquired in falling from

the centre B to the periphery $D\,F\,E$. Therefore each of the balls would describe the elliptic arc $D\,F\,E$ around B as a centre (Newton's Principia, Book 1st, Prop. 10, Cor. 1st), and they would meet at F, in the extremity of the line $B\,F$, which bisects the area $D\,F\,E\,B$ (Principia, Book 1st, Prop. 1st).

2d. Suppose the balls begin to fall at the same instant at A and C, then it may be shown by reasoning entirely similar to the above that the two balls will meet somewhere in the elliptic arc $D\,F\,E$.

To show at what point they will meet, let us with radii $B\,D$ and $B\,C$ describe the two circular quadrants $D\,S\,H$ and $C\,G\,A$, and let us suppose balls projected from the points C and D, with a force and direction requisite to cause them to describe the curves $D\,F\,E$, $D\,S\,H$, and $C\,G\,A$.

Now it is a well known principle in dynamics that the times of describing these arcs are all equal to each other. A ball in its passage along the arc $C\,G\,A$ would occupy the same time in reaching the point G, as the ball which passes from C to B would occupy in reaching D through the versed sine $C\,D$ of the arc $C\,G$, and in passing from G to A it would occupy the same time as the ball in its passage from D to B.

Hence the time of describing the arc $C\,G\,A$ is equal to the time of falling from C to B along the line $C\,B$. It is further evident that the time occupied from the beginning of motion at A and C till the two balls meet in the arc $D\,F\,E$ will be the same as either of them would occupy in passing from A or C to B, along the lines $A\,B$ and $C\,B$.

Therefore the time of passing through the arc of the quadrant $C\,G\,A$ is at once the measure of the time of passage through the arcs $D\,S\,H$ and $D\,F\,E$, and that of the balls in their passage from C to B and A to B, along the lines $C\,B$ and $A\,B$, and also that of the time from the beginning of motion at A and C, till the time of meeting in the elliptic arc $D\,F\,E$.

Hence if we bisect the arc $D\,S\,H$ or the angle $D\,B\,E$ by the line $B\,N$, the point I where this line cuts the arc of the ellipse will be the point where the two balls will meet.

What then will become of the balls?

The answer to this question is now manifest.

On striking their respective planes at D and E the balls will be deflected in directions perpendicular to $A\,B$ and $C\,B$, and immediately

begin their motion in the elliptic orbit *D F E*, and continue till they meet at *F*, or *I*, when, being equal, and meeting with equal velocity, and being perfectly elastic, they will return by the same trajectory, and with the same velocity as that with which they met, striking their respective planes at *D* and *E* at the same instant, where they will be deflected in the directions *E A* and *D C*, and arrive finally at *A* and *C* at the same instant.

Their whole force in those directions being now overcome, they will immediately begin to fall again through the same path, striking the planes, and describing the elliptic orbit, and meeting as before, and thus will they continue to describe and re-describe the same path, advancing and repelling each other forever.

BOOK NOTICES.

We have received from the publisher, *James S. Burnton*, 149 Grand street, New York, "*My Visit to the Sun; or, Critical Essay on Physics, Metaphysics and Ethics.*" By Lawrence S. Benson, author of "Benson's Geometry."

This book is an octavo vol. of 164 pages, and is executed on material, and in a style which is highly creditable to the publisher.

The author, no doubt, *intends* to represent, in this fictitious dialogue between himself and an inhabitant of the sun, by his own discourses, what he believes to be the current scientific views upon the subjects discussed, and by the discourses of the inhabitant of the sun, his own views upon the same subjects.

Anything like a critical review of the book would be absurd. It is sufficient to say that throughout the 164 pages we have not been able to find a single lucid paragraph; and that in the author's discourses, in which he has intended to represent the current scientific views on the subjects under discussion, he has invariably misrepresented or distorted science.

PAMPHLETS RECEIVED.

A Method of Computing Absolute Perturbations, by G. W. Hill, office of the American Ephemeris, Washington, D. C. 8 pages quarto. From "*Astronomische Nachrichten*," No. 1982.

A Method of Integrating the Square Roots of Quadratics, by Henry T. Eddy, C. E., Ph. D. 16 pages octavo. From the proceedings of the University Convocation, held at Albany, N. Y., August 6th, 7th & 8th, 1872.

Discussion of the Applied Forces in a Draw-Bridge, by H. T. Eddy, C. E., Princeton, N. J. 8 pages octavo. From *Van Nostrand's Eclectic Engineering Magazine* for June, 1874.

Annual Register of the Rensselaer Polytechnic Institute, Troy, N. Y., 1873-4.

Catalogue of North Carolina College, Mt. Pleasant, N. C., 1873-4.

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EDUCATIONAL TESTIMONY CONCERNING THE CALCULUS.

BY LEVI W. MEECH, A. M., HARTFORD, CONN.

The following notes refer primarily to persons studying the Calculus without the advantage of an experienced instructor.

1. An eminent teacher long familiar with Algebra and Geometry writes : "My efforts to learn the Calculus, for several years, were entirely unsuccessful. I had tried three different authors, without making any progress. Lately I procured a fourth treatise, which has given me quite an insight, so that I begin to enjoy the study. Most of my leisure will now be given to this favorite treatise."

2. Another similar proficient in Algebra and Geometry says : "I procured a work on the Calculus some months ago, intending to master it ; but there is something so obscure in the first part, that I cannot understand it. The rules for differentiating are plain enough. I can differentiate, but I don't understand what it signifies."

3. While writing this, a lady teacher calls to say that her "class in Geometry has been at a stand, for more than a week, over the proposition that a circle is a polygon of an infinite number of sides. Can the contour of a polygon with its straight lines and angles ever be a circle ; or can a circle be a polygon ?" Of course, it is the difficulty of conceiving how the small difference between the inscribed and the circumscribed polygon entirely vanishes.

4. A cadet at West Point, extremely fond of mathematics, thus estimates the Calculus : "The inventors of the Differential and Integral Calculus have claimed that this branch of so-called science belongs to the department of mathematics ; and laboring under that delusion, have introduced it into the course of academical instruction for the torture of

students. Such classification is obviously incorrect, because the principles of mathematics fall within the scope of the reasoning faculty. The Calculus, on the contrary, lies without the boundaries of reason." (Life of Gen. Nathaniel Lyon, p. 30.)

5. Bishop Berkley in 1734: "I have no controversy about your conclusions, but only about your logic; and it must be remembered that I am not concerned about the truth of your theorems, but only about the way of coming at them."

Leaving the collection of further abundant testimony to some educational commission, we find the modern treatises to be, in general, extremely well written. The discouragement and demoralization of learners appear to be caused by two or three pages only, near the beginning, which describe what is termed, "passing to the limit." In this, if our examination be not mistaken, the suppressed premise of the logic consists in a misconception of the quantity usually denoted by $u' - u$. Under previous training, the student naturally regards it as a homogeneous magnitude, whereas the principles of logic require him to regard it here *as a graded series*, like the right hand member of the equation.

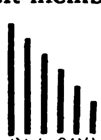
Let the common operation of the student's mind be attentively observed. In resolving quadratic equations, he has been accustomed to regard the similar quantity as a single magnitude, and very naturally he still so regards it; thus, in the ordinary course,

$$u' - u = ah + bh^2, \text{ or}$$
$$\rule{2cm}{2pt} = ah + bh^2.$$

And when the equation is divided by h, his idea is still that of a single magnitude in the quotient; thus

$$\frac{\rule{2cm}{2pt}}{h} = a + bh.$$

Consequently, the passage to the limit or decreasing h to zero, has some effect upon the left member, to him incomprehensible; thus,



$$000\,000\frac{0}{0} = a + b \times 0.$$

Again and again the learner attempts to form a clear conception of the effect in the left hand member, and more often falters and recoils in the useless effort. Is it not like soldiers repeatedly attempting to storm

an impregnable redoubt, till they become demoralized? Nor should it be forgotten that ordinary learners have never had the previous training in Wallis's Arithmetic of Infinites and in the subtle metaphysics of the middle ages, which Newton and Leibnitz had enjoyed.

But what is the remedy? Our answer would be, primarily, to regard the quantity $u' - u$ as a graded series, like the right hand member; and secondly, to substitute such graded series always in place of $u' - u$, before passing to the limit, or in the language of Newton, "equating the homologous terms of the resulting equation."

"This Calculus," says Laplace, "is only the comparison of the coefficients of the same powers of the differentials or increments, in the development in series, of functions identically equal." (Essai Philosophique sur les Probabilites, p. 56).

In this aspect, the first principles of the Calculus may be correctly derived from the method of indeterminate coefficients. Indeed, some of the simpler applications are already introduced into Algebra under the name of "derived polynomials," or other designations. It were a desirable improvement in works of Algebra, after Indeterminate Coefficients to devote a few pages to the foundation of the Calculus, in connection with the simple notation of Leibnitz, now almost universally acceptable. This would be immediately useful in unfolding the theory of the higher equations, besides other important advantages. For illustration, let the expansion of the binomial $(x + h)$ be denoted by the following notation:

$$(x+h)^n = x^n + nx^{n-1}h + n(n-1)x^{n-2}\frac{h^2}{1.2} + n(n-1)(n-2) \times x^{n-3}\frac{h^3}{1.2.3} + \dots$$

$$(x+h)^n = fx + f'x.h + f''x.\frac{h^2}{1.2} + f'''x.\frac{h^3}{1.2.3} + \dots$$

$$u' = u + \frac{du}{dx}.h + \frac{d^2u}{dx^2}.\frac{h^2}{1.2} + \frac{d^3u}{dx^3}.\frac{h^3}{1.2.3} + \dots$$

Any algebraic expression containing x is termed a function of x, and may be denoted by fx, or by the single letter u. In the foregoing equations, $u = fx = x^n$ is the primitive function. Also $f'x = nx^{n-1}$ is termed the *first* derivative function; $f''x = n(n-1)x^{n-2}$ is termed the *second* derivative function, and so on. The first derivative is more usually denoted by the ratio

$$\frac{du}{dx},$$

where du and dx denote two undetermined increments, named differential of x, differential of u. The second derivative is the ratio of the second differential of u divided by the square of the differential of x; and so on. It should be observed that $f'x$ or

$$\frac{du}{dx},$$

etc., so far correspond entirely to quantities heretofore denoted by A, B, etc., in the common algebraic operations of Indeterminate Coefficients.

This notation is not restricted to the simple case of the binomial just described. Let $u = fx$ denote any function whether a single term or a polynomial, radical or fractional. And let $u' = f(x + h)$ denote its value when x receives the increment or decrement h. Now suppose the function to be expanded by the binomial theorem aided by algebraic division or multiplication, into an ascending series of the powers of h. Since h is an undetermined increment we may suppose it to be so small that the series will be convergent. Denoting the coeffiicients by the same notation as before, we have the augmented function,

$$u' = u + \frac{du}{dx}.h + \frac{d^2u}{dx^2}.\frac{h^2}{1.2} + \frac{d^3u}{dx^3}.\frac{h^3}{1.2.3} + \ldots.$$

This admirable extension of the binomial formula is called from Dr. Brook Taylor, its author, Taylor's Theorem of increments. So far as the coefficients of the powers of h denote mere quantity, no further demonstration is needed ; but they were intended to denote their mode of derivation also. Here now is the graded series for establishing the Calculus with all the rigorous accuracy of Algebra. For example, let $u = ax^2 + bx + c$.

Let u' denote the augmented value when x received the undetermined increment h.

$$u' = ax^2 + bx + c + (2ax + b)h + ah^2,$$
$$u' - u = (2ax + b)h + ah^2.$$

Substituting in place of $u' - u$ the series of Taylor's Theorem,

$$\frac{du}{dx}.h + \frac{d^2u}{dx^2}.\frac{h^2}{2} + \ldots. = (2ax + b)h + ah^2.$$

Or dividing through by h,

$$\frac{du}{dx}+\frac{d^2u}{dx^2}\cdot\frac{h}{2}+\ldots\ldots=2ax+b+ah.$$

"Passing to the limit," that is making h to be 0,

$$\frac{du}{dx}=2ax+b.$$

In this manner, the usual difficulty is entirely avoided. In indicating this modification, however, we have followed the popular notion that learners of the Calculus must first attend to some characteristic demonstration. But really, on reconsidering the question, what does it amount to? We have denoted the indeterminate coefficient of h by the symbol

$$\frac{du}{dx},$$

have gone through the usual course of demonstration, and—reproduced the original definition. "Passing to the limit" has amounted to nothing more.

The important inference hence arises, that a mere definition of the derivative

$$\frac{du}{dx},$$

as the coefficient of h, will suffice at the beginning, without that most unfortunate initiation of "passing to the limit." We can therefore commend to writers of Algebra the insertion of the formal notation and first simple rules of the Calculus, as an extension of the method of indeterminate coefficients, with still more satisfaction and confidence.

One further remark,—most writers have preferred the "infinitesimals" of Leibnitz to the "flowing quantities" of Newton. But at the first, leaving these to future applications, will it not be simpler to regard the quantities as primitive and derivative only, like the binomial coefficients or terms? As some ingenious writer has remarked, the increments need only be so small, that to suppose them smaller would not change the character of the results.

INTEGRATION OF POLYNOMIAL DIFFERENTIALS—GENERAL FORMULAE OF REDUCTION.

BY PROF. DASCOM GREENE, TROY, N. Y.

Reduction Formulae, whereby the integration of a given differential is made to depend on that of another of a more simple form, play an important part in the Integral Calculus. They are usually obtained by the

application of the formula for integration by parts, which is itself nothing more than a formula of reduction. This process, however, applied to polynomial differentials of more than two terms, is so tedious that the formulae for higher polynomials are seldom given in the text books.

In this paper I have established by a method of remarkable simplicity, general reductions which include the reduction formulae for all polynomial differentials, and from which those answering to a given polynomial can be immediately obtained by making the proper substitutions.

Let X represent any polynomial, that is,

(a) $\quad X = ax^h + bx^n + cx^r + \text{\&c.},$

in which a, b, c, h, n, r, &c. are any constants, then

$$dX = (hax^{h-1} + nbx^{n-1} + rcx^{r-1} + \text{\&c.})dx,$$

and we have

$$d(x^m X^p) = x^{m-1} X^{p-1}(mXdx + pxdX)$$

(b) $$= x^{m-1}X^{p-1}[m(ax^h + bx^n + cx^r + \text{\&c.})dx + p(hax^h + nbx^n + rcx^r + \text{\&c.})dx]$$

$$= x^{m-1}X^{p-1}[(m + ph)ax^h dx + (m + pn)bx^n dx + (m + pr)cx^r dx + \text{\&c.}]$$

or $$x^m X^p = a(m + ph)\int x^{m+h-1}dxX^{p-1} + b(m + pn)\int x^{m+n-1}dxX^{p-1}$$

(1). $$+ c(m + pr)\int x^{m+r-1}dxX^{p-1} + \text{\&c.}$$

Resuming now equation (b), and eliminating ax^h by means of (a), we have

$$d(x^m X^p) = x^{m-1}X^{p-1}[mXdx + ph(X - bx^n - cx^r - \text{\&c.})dx + pnbx^n dx + \text{\&c.}]$$

$$= x^{m-1}X^{p-1}[(m + ph)Xdx + p(n - h)bx^n dx + p(r - h)cx^r dx + \text{\&c.}]$$

or $$x^m X^p = (m + ph)\int x^{m-1}dxX^p + bp(n - h)\int x^{m+n-1}dxX^{p-1}$$

(2) $$+ cp(r - h)\int x^{m+r-1}dxX^{p-1} + \text{\&c.}$$

In a similar way, eliminating bx^n, cx^r, &c., successively, we shall find

$$x^m X^p = (m + pn)\int x^{m-1}dxX^p + ap(h - n)\int x^{m+h-1}dxX^{p-1}$$

(3) $$+ cp(r - n)\int x^{m+r-1}dxX^{p-1} + \text{\&c.}$$

$$x^m X^p = (m + pr)\int x^{m-1}dx X^p + ap(h - r)\int x^{m+h-1}\,dx X^{p-1}$$

$$(4) \qquad + bp(n - r)\int x^{m+n-1}dx X^{p-1} + \&c.$$

$$\&c. \qquad\qquad \&c.$$

Equating the second members of (1) and (2), we have the identical equation

$$a(m + ph)\int x^{m+h-1}dx X^{p-1} + b(m + ph)\int x^{m+n-1}dx X^{p-1}$$

$$+ c(m + ph)\int x^{m+r-1}dx X^{p-1} + \&c. = (m + ph)\int x^{m-1}dx X^p$$

$$\text{or} \int x^{m-1}dx X^p = a\int x^{m+h-1}dx X^{p-1} + b\int x^{m+n-1}dx X^{p-1}$$

$$(5) \qquad + c\int x^{m+r-1}dx X^{p-1} + \&c.$$

These elegant symmetrical formulae are likewise entirely general, being true for all values of m, p, h, n, r, &c., positive or negative, entire or fractional. Two or three examples will show the facility with which they may be used in deriving formulae of reduction.

I. If we limit the number of terms of the polynomial to *three*, and make $h = 0$, and $r = 2n$, then

$$X = a + bx^n + cx^{2n},$$

and (1), (2), (3) and (4) reduce to

$$x^m X^p = am\int x^{m-1}dx X^{p-1} + b(m + pn)\int x^{m+n-1}dx X^{p-1}$$

$$(6) \qquad + c(m + 2pn)\int x^{m+2n-1}dx X^{p-1}$$

$$x^m X^p = m\int x^{m-1}dx X^p + bpn\int x^{m+n-1}dx X^{p-1}$$

$$(7) \qquad + 2cpn\int x^{m+2n-1}\,dx X^{p-1}$$

$$x^m X^p = (m + pn)\int x^{m-1}dx X^p - apn\int x^{m-1}dx X^{p-1}$$

$$(8) \qquad + cpn\int x^{m+2n-1}dx X^{p-1}$$

$$x^m Xp = (m + 2pn)\int x^{m-1}\,dx X^p - 2apn\int x^{m-1}dx X^{p-1}$$

$$(9) \qquad - bpn\int x^{m+n-1}dx X^{p-1}$$

Each of the equations (6), (7), (8) and (9), expresses a relation between three integrals, and being solved for each in succession, will furnish three formulae for the reduction of the integral of the trinomial differential $x^m dx(a+bx^n+cx^{2n})^p$. It is unnecessary to develope them in detail here.

II. If $n = 1$, the equations just obtained reduce to

$$X = a + bx + cx^2,$$

$$(10)\quad x^m X^p = am\int x^{m-1}dxX^{p-1} + b(m+p)\int x^m dxX^{p-1} + c(m+2p)\int x^{m+1}dxX^{p-1}$$

$$(11)\quad x^m X^p = m\int x^{m-1}dxX^p + bp\int x^m dxX^{p-1} + 2cp\int x^{m+1}dxX^{p-1}$$

$$(12)\quad x^m X^p = (m+p)\int x^{m-1}dxX^p - ap\int x^{m-1}dxX^{p-1} + cp\int x^{m+1}dxX^{p-1}$$

$$(13)\quad x^m X^p = (m+2p)\int x^{m-1}dxX^p - 2ap\int x^{m-1}dxX^{p-1} - bp\int x^m dxX^{p-1}$$

Each of these equations will give three reduction formulae for

$$\int x^m dx(a + bx + cx^2)^p.$$

III. If $c = 0$, equations (6), (7) and (8) become

$$(14)\qquad x^m X^p = am\int x^{m-1}dxX^{p-1} + b(m+pn)\int x^{m+n-1}dxX^{p-1}$$

$$(15)\qquad x^m X^p = m\int x^{m-1}dxX^p + bpn\int x^{m+n-1}dxX^{p-1}$$

$$(16)\qquad x^m X^p = (m+pn)\int x^{m-1}dxX^p - apn\int x^{m-1}dxX^{p-1}$$

in which $X = a + bx^n$.

Solving (14), (15) and (16) for each integral in succession, and writing the results in their simplest form, we have the following well-known formulae for the reduction of binomial differentials :

$$(17)\qquad \int x^{m-1}dxX^p = \frac{x^m X^{p+1}}{am} - \frac{b(m+pn+n)}{am}\int x^{m+n-1}dxX^p$$

(18) $$\int x^{m-1}dxX^p = \frac{x^{m-n}X^{p+1}}{b(m+pn)} - \frac{a(m-n)}{b(m+pn)}\int x^{m-n-1}dxX^p$$

(19) $$\int x^{m-1}dxX^p = \frac{x^mX^p}{m} - \frac{bpn}{m}\int x^{m+n-1}dxX^{p-1}$$

(20) $$\int x^{m-1}dxX^p = \frac{x^{m-n}X^{p+1}}{bn(p+1)} - \frac{m-n}{bn(p+1)}\int x^{m-n-1}dxX^{p+1}$$

(21) $$\int x^{m-1}dxX^p = \frac{x^mX^p}{m+pn} + \frac{apn}{m+pn}\int x^{m-1}dxX^{p-1}$$

(22) $$\int x^{m-1}dxX^p = -\frac{x^mX^{p+1}}{an(p+1)} + \frac{m+pn+n}{an(p+1)}\int x^{m-1}dxX^{p+1}.$$

It should also be observed that equations (14), (15) and (16) may be obtained directly by following the general method by which (1) and (2) were derived, and this method will be found much more simple than that generally pursued in obtaining reduction formulae for binomials.*

**Erratum*. On page 138, line 6, for "general reductions" read general relations.

USEFUL FORMULAE IN THE CALCULUS OF FINITE DIFFERENCES.

BY G. W. HILL.

The finding of the values of the differential coefficients of a function of a single variable and of the single and double integrals with respect to the independent variable, from special values of the function computed at equidistant intervals, is an operation very frequent in Planetary Astronomy. The following seems a simpler exposition of the matter than has hitherto been given :

Let y be a function of x computed for the series of values of x, ... $a-h$, a, $a+h$, $a+2h$, ...; and let the differences and first and second summed values of y be denoted thus,

....	....	....	....	....	....	....	....
$a-h$	$\Delta^{-2}y_{-1}$	$\Delta^{-1}y_{-\frac{3}{2}}$	y_{-1}	$\Delta y_{-\frac{3}{2}}$	$\Delta^2 y_{-1}$	$\Delta^3 y_{-\frac{3}{2}}$	....
a	$\Delta^{-2}y_0$	$\Delta^{-1}y_{-\frac{1}{2}}$	y_0	$\Delta y_{-\frac{1}{2}}$	$\Delta^2 y_0$	$\Delta^3 y_{-\frac{1}{2}}$	....
$a+h$	$\Delta^{-2}y_1$	$\Delta^{-1}y_{\frac{1}{2}}$	y_1	$\Delta y_{\frac{1}{2}}$	$\Delta^2 y_1$	$\Delta^3 y_{\frac{1}{2}}$	....
$a+2h$	$\Delta^{-2}y_2$	$\Delta^{-1}y_{\frac{3}{2}}$	y_2	$\Delta y_{\frac{3}{2}}$	$\Delta^2 y_2$	$\Delta^3 y_{\frac{3}{2}}$	....
....	....	....	....	....	....	....	....

With regard to the differences of odd orders let us adopt the general notation

$$\Delta^{2n+1}y_i = \tfrac{1}{2}\left(\Delta^{2n+1}y_{i-\frac{1}{2}} + \Delta^{2n+1}y_{i+\frac{1}{2}}\right),$$

n and i being integers. In this way the symbol Δ does not follow the law of indices as in the ordinary method of differences, that is we do not have in general

$$\Delta^n\Delta^{n'} = \Delta^{n+n'}.$$

Nevertheless it is evident the following relations hold:

$$\Delta^{2n}\Delta^{2n'} = \Delta^{2(n+n')}, \quad \Delta^{2n+1}\Delta^{2n'} = \Delta^{2(n+n')+1},$$

that is, the exponents are to be added except when both are odd.

For brevity writing D for $\frac{d}{dx}$, and e denoting the base of hyperbolic logarithms, the symbolical expression for Taylor's Theorem gives

$$y_{-1} = e^{-hD}y_0, \quad y_0 = y_0, y_1 = e^{hD}y_0.$$

Whence it is easy to see that

$$\Delta = \frac{e^{hD} - e^{-hD}}{2}, \quad \Delta^2 = e^{hD} + e^{-hD} - 2.$$

The last may be written

$$\Delta^2 = \left(e^{\frac{hD}{2}} - e^{-\frac{hD}{2}}\right)^2.$$

Thus it is evident that we have in general,

$$\Delta^{2n} = \left(e^{\frac{hD}{2}} - e^{-\frac{hD}{2}}\right)^{2n}, \quad \Delta^{2n+1} = \left(e^{\frac{hD}{2}} - e^{-\frac{hD}{2}}\right)^{2n+1} \cdot \frac{e^{\frac{hD}{2}} + e^{-\frac{hD}{2}}}{2}.$$

Also

$$e^{\frac{hD}{2}} - e^{-\frac{hD}{2}} = \sqrt{\Delta^2}, \quad e^{\frac{hD}{2}} + e^{-\frac{hD}{2}} = \sqrt{4 + \Delta^2}.$$

Whence

$$h^2D^2 = 4\log^2\left(\frac{\Delta}{2} + \sqrt{1 + \tfrac{1}{4}\Delta^2}\right) = \left(\int \frac{d\Delta}{\sqrt{1 + \tfrac{1}{4}\Delta^2}}\right)^2,$$

the integral being taken so as to vanish with Δ. The operation denoted by the symbolical expression $\frac{\Delta}{D}$ is evidently a function of Δ^2, and we have

$$\frac{\Delta}{hD} = \left(e^{\frac{hD}{2}} - e^{-\frac{hD}{2}}\right) \cdot \frac{e^{\frac{hD}{2}} + e^{-\frac{hD}{2}}}{2hD} = \frac{\Delta\sqrt{1+\frac{1}{4}\Delta^2}}{hD}$$

$$= \Delta\sqrt{1+\tfrac{1}{4}\Delta^2}\left(\int\frac{d\Delta}{\sqrt{1+\frac{1}{4}\Delta^2}}\right)^{-1}.$$

Whence

$$hD = \frac{1}{\sqrt{1+\frac{1}{4}\Delta^2}}\int\frac{d\Delta}{\sqrt{1+\frac{1}{4}\Delta^2}}.$$

It is plain, then, in general, that we have the value of an even differential coefficient from the formula

$$D^{2n} = h^{-2n}\left(\int\frac{d\Delta}{\sqrt{1+\frac{1}{4}\Delta^2}}\right)^{2n},$$

and the value of an odd one from

$$D^{2n+1} = \frac{h^{-2n-1}}{\sqrt{1+\frac{1}{4}\Delta^2}}\left(\int\frac{d\Delta}{\sqrt{1+\frac{1}{4}\Delta^2}}\right)^{2n+1}.$$

Differentiating the first of these with respect to Δ, we obtain

$$\frac{d.D^{2n}}{d\Delta} = \frac{2nh^{-2n}}{\sqrt{1+\frac{1}{4}\Delta^2}}\left(\int\frac{d\Delta}{\sqrt{1+\frac{1}{4}\Delta^2}}\right)^{2n-1} = 2nh^{-1}D^{2k-1}.$$

Thus the value of an even differential coefficient can be obtained from that of the preceding differential coefficient by the very simple formula

$$D^{2n} = 2nh^{-1}\int D^{2n-1}d\Delta.$$

If we employed the preceding formula for D for expanding its value in powers of Δ, we should find it difficult to discover the law of the numerical coefficients, but by differentiating the value of D^{2n+1} with respect to Δ, we shall find that it satisfies the differential equation

$$[1+\tfrac{1}{4}\Delta^2]\frac{dD^{2n+1}}{d\Delta} + \frac{\Delta}{4}D^{2n+1} = (2n+1)h^{-1}D^{2n},$$

which when $n = 0$, becomes

$$[1+\tfrac{1}{4}\Delta^2]\frac{dD}{d\Delta} + \frac{\Delta}{4}D = h^{-1}.$$

If in this we suppose

$$D = h^{-1}\Sigma.A_n\Delta^n,$$

we shall find that the coefficients A_n satisfy in general the relation

$$(n+2)A_{n+2} + \frac{n+1}{4}A_n = 0,$$

whence

$$A_{n+2} = -\frac{n+1}{4(n+2)}A_n.$$

And as we know that $A_1 = 1$, this suffices for obtaining all the coefficients in succession. In the general case if we put

$$D^{2n} = h^{-2n}\Sigma.A_i^{(2n)}\Delta^i, \quad D^{2n+1} = h^{-2n-1}.\Sigma A_i^{(2n+1)}\Delta^i,$$

the above differential equation gives the following relation between the coefficients:

$$A_{i+2}^{(2n+1)} = -\frac{i+1}{4(i+2)}A_i^{(2n+1)} + \frac{2n+1}{i+2}A_{i+1}^{(2n)},$$

by which all the coefficients in succession may be derived.

We have

$$D = h^{-1}\left(\Delta - \frac{1}{3}\,\frac{\Delta^3}{2} + \frac{1.2}{3.5}\,\frac{\Delta^5}{2^2} - \frac{1.2.3}{3.5.7}\,\frac{\Delta^7}{2^3} + \dots\right),$$

$$D^2 = h^{-2}\left(\Delta^2 - \frac{1}{3.2}\,\frac{\Delta^4}{2} + \frac{1.2}{3.5.3}\,\frac{\Delta^6}{2^2} - \frac{1.2.3}{3.5.7.4}\,\frac{\Delta^8}{2^3} + \dots\right),$$

where the law of the coefficients is readily seen. In the higher differential coefficients the fractions are more complex, we therefore content ourselves with writing the values thus,

$$D^3 = h^{-3}\left(\Delta^3 - \frac{1}{4}\Delta^5 + \frac{7}{120}\Delta^7 - \frac{41}{1024}\Delta^9 + \dots\right),$$

$$D^4 = h^{-4}\left(\Delta^4 - \frac{1}{6}\Delta^6 + \frac{7}{240}\Delta^8 - \frac{41}{7560}\Delta^{10} + \dots\right),$$

$$D^5 = h^{-5}\left(\Delta^5 - \frac{1}{3}\Delta^7 + \frac{13}{144}\Delta^9 - \dots\dots\dots\dots\right),$$

$$D^6 = h^{-6}\left(\Delta^6 - \frac{1}{4}\Delta^8 + \frac{13}{240}\Delta^{10} - \dots\dots\dots\dots\right).$$

The above expressions for D^{2n} and D^{2n+1} are equally applicable when n is negative; they then give the formulae to be used in mechanical quadratures, thus

$$D^{-1} = \frac{h}{\sqrt{1+\frac{1}{4}\Delta^2}}\left(\int\frac{d\Delta}{\sqrt{1+\frac{1}{4}\Delta^2}}\right)^{-1},$$

$$D^{-2} = h^2\left(\int\frac{d\Delta}{\sqrt{1+\frac{1}{4}\Delta^2}}\right)^{-2}.$$

If these expressions are expanded in powers of Δ, we obtain

$$D^{-1} = h\left(\Delta^{-1} - \frac{1}{12}\Delta + \frac{11}{720}\Delta^3 - \frac{191}{60480}\Delta^5 + \frac{2497}{3628800}\Delta^7 - \frac{14797}{95800320}\Delta^9 + \frac{92427157}{2615348736000}\Delta^{11} - \dots\dots\right),$$

$$D^{-2} = h^2\left(\Delta^{-2} + \frac{1}{12}\Delta^0 - \frac{1}{240}\Delta^2 + \frac{31}{60480}\Delta^4 - \frac{289}{3628800}\Delta^6\right.$$

$$+ \frac{317}{22809600} \Delta^8 - \ldots\ldots\ldots\ldots\ldots\ldots\Big).$$

These are the expressions to be used in computing the values of the integrals $\int y dx$ and $\int\int y dx^2$. It must be noticed that Δ^{-1} virtually contains an arbitrary constant C, and Δ^{-2} an arbitrary expression $Cx + C'$. In fact the quantities in the columns to the left of that of the function y cannot be written until we know one quantity in each column. These constants C and C' are usually determined from the given values of $\int y dx$ and $\int\int y dx^2$ for $x = a$. If we denote them by D_0^{-1} and D_0^{-2} and if in general the subscript ($_0$) denote values which obtain when $x = a$, it will be seen that

$$\Delta_0^{-1} = \frac{D_0^{-1}}{h} + \frac{1}{12}\Delta_0 - \frac{11}{720}\Delta_0^3 + \ldots\ldots$$

$$\Delta_0^{-2} = \frac{D_0^{-2}}{h^2} - \frac{1}{12}\Delta_0^0 + \frac{1}{240}\Delta_0^2 - \ldots\ldots$$

Having thus the sum and difference of the quantities $\Delta^{-1}y_{-\frac{1}{2}}$ and $\Delta^{-1}y_{\frac{1}{2}}$, it will be easy to get the quantities themselves.

In using the method of mechanical quadratures it is usual to multiply the values of y by h, if the single integral only is wanted, but by h^2 if the double is also to be obtained; in the last case then it is necessary to divide the results obtained by h in order to have the single integral.

These formulae appear to have been first obtained by Gauss (*Werke*, Vol. III, p. 328). Encke has given them in the Berlin *Jahrbuch* for 1838. For use they are much superior to the formula given by Laplace (*Mecanique Celeste*, Vol. IV, p. 207).

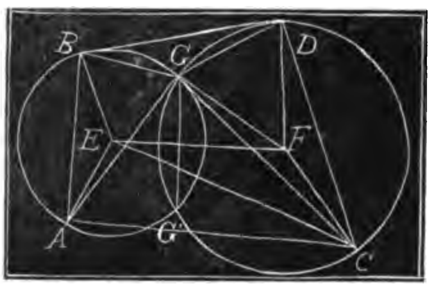
PROBLEM IN "CURVES OF PURSUIT," BY WERNER STILLE, MARINE, ILL.—A hunter is in the field with his dog. The hunter, standing at a point A calls his dog, which is at a point B. At this moment the hunter walks off in a direction right-angular to AB, while the dog, running n times as fast as his master walks, keeps his master continually in view, thus at each moment running along the right line which then lies between him and his master.

Find thn Curve along which the dog runs until he overtakes his master at C.

THE FOUR-POINT PROBLEM.

BY T. J. LOWRY, U. S. C. S., SAN FRANCISCO, CAL.

PROBLEM: Four points in the same plane being given in position, to determine the position of any other point (or place of observation) in reference to these given points, having on the four points two angles which have no parts common except their vertices.



Let A, B, D, & C be the four given points: and let DGC & AGB be the observed angles. And then since the $\angle AEB = 2\angle AGB$, and $AE = BE$ ∴ in $\triangle AEB$ we have BA and all the $\angle$s to find AE. And then in $\triangle$ EAC are known AE and AC and $\angle EAC$ to find EC and the $\angle$ ECA. But since $\angle DFC = 2\angle DGC$, and DF=FC hence in $\triangle$ DFC we have DC and all the $\angle$s to get FC. But the $\angle FCE = \angle ACD - > ACE - > FCD$ ∴ in $\triangle FCE$ are given FC and CE and the included angle to find FE and $>EFC$, then in $\triangle$ EGF all the sides are known, hence $>GFE$ can be found. But $>GFC = >EFC + >GFE$ ∴ in $\triangle$ GFC we have GF and FC and all the angles to find GC. Then in $\triangle$ GCD are given GC, CD and $>CGD$ to get GD. And now GA and GB follow in a very obvious manner.

This problem may obviously be solved with fewer equations by using quadrilaterals, instead of triangles, in the solution.

Unless the two circles of position are tangent to each other there will be two points (viz, G and G′) which will equally well satisfy the conditions of the problem: hence in this case we judge from an approximate knowledge of our position which of the points G or G′ we were at when we observed.

The manner of sweeping the circles of position, with Protractor and Dividers is easy and expeditious: since $>AEB = 2>AGB$ then $>ABE$

$$= \frac{180^\circ - >AEB}{2} = \frac{180^\circ - 2>AGB}{2} = 90^\circ - AGB,$$

that is $>ABE$ = the complement of the observed $>AGB$. The rule for plotting the circles of position is now obviously, to lay off from AB at the points A, and B the compliment of the observed angle AGB, and the point of intersection of the produced sides of these angles will be

the center of a circle of position. And now from this point as center with radius AE (or BE) sweep the circle of position. And in like manner lay down the other circle of position through C and D.

This problem will often be found servicable to the Hydrographer and Explorer when from either accident or necessity only two angles are measured on four objects.

SOLUTIONS OF PROBLEMS IN NO. 6.

Solutions of problems in No. 6 have been received as follows: From Geo. L. Dake, 25 & 26; R. M. DeFrance, 25 & 26; Prof. A. B. Evans, 25, 26, 27 & 29; Henry Gunder, 25, 26, 27 & 29; Wm. Hoover, 26; Prof. A. Hall, 29; H. Heaton, 29; D. J. McAdam, 25 & 26; Esther Matthews, 26; Artemas Martin, 27 & 29; A. W. Phillips, 25, 26, 27 & 29; L. Regan, 25, 26 & 29; R. L. Selden, 25; Werner Stille, 25, 26, 27 & 29; E. B. Seitz, 25, 26, 27 & 29; Prof. J. Scheffer, 26 & 29; Walter Siverly, 27.

25. "Required the sides of an obtuse angled triangle the area of which is 14.048 acres, the obtuse angle 111°15′, and one of the acute angles 11°44′10″."

SOLUTION BY HENRY GUNDER, GREENVILLE, OHIO.

Putting $A = 111°15'$, $B = 11°44'10''$, $C = 57°50''$, and x, y, z or the sides opposite A, B and C, and $a = 14.048$ acres $= 2247.68$ sq. rods.

Since the product of two sides and the sine of the included angle equals twice the area we have,

$$(1)\ xy = \frac{2a}{\sin C},\quad (2)\ xz = \frac{2a}{\sin B},\quad (3)\ yz = \frac{2a}{\sin A}.\quad \text{Then}$$

$$\sqrt{\frac{(1)\times(2)}{(3)}} = (4)\ x = \sqrt{\frac{2a \sin A}{\sin B \sin C}}.\quad \text{Similarly } y = \sqrt{\frac{2a \sin B}{\sin A \sin C}}.$$

$$\text{“}\quad z = \sqrt{\frac{2a \sin C}{\sin A \sin B}}.$$

By applying logarithms, $x = 156.705$ rods, $y = 34.1997$ rods, $z = 141.034$ rods.

26. "Find θ from the equation $15 \sin \theta + 12 \cos \theta = 17.97240$, (1)."

SOLUTION BY WILLIAM HOOVER, SOUTH BEND, IND.

The given equation may be written, $m \sin \theta + n \cos \theta = q$......(2). In (2) put $p \cos \psi = m$ and $p \sin \psi = n$ and it becomes

$$\sin(\theta + \psi) = \frac{q}{p} \quad \text{.........................(3).}$$

$$\text{But } \frac{p \sin \psi}{p \cos \psi} = \tan \psi = \frac{n}{m}. \quad \therefore \psi = 38^\circ 39' 35''.$$

$$\theta + \psi = \sin^{-1}\frac{q}{n} \sin \psi = 69^\circ 19' 35''. \qquad \therefore \theta = 30^\circ 40',$$

---

27. "Four given equal spheres being placed in close contact with each other, it is required to find the volume of the space inclosed between them and the four triangular planes drawn respectively through each three centers."

SOLUTION BY E. B. SEITZ, GREENVILLE, O.

Let r = the radius of each sphere. Then $\frac{2}{3}r^3\sqrt{2}$ = the volume of the tetraedron formed by the four planes, and $\frac{4}{3}(3\cos^{-1}\frac{1}{3} - \pi)r^3$ = the volume of the four equal spherical sectors cut from the spheres by the planes.

Hence, the required volume is

$$V = \tfrac{2}{3}r^3\sqrt{2} - \tfrac{4}{3}(3\cos^{-1}\tfrac{1}{3} - \pi)r^3 = \tfrac{2}{3}r^3(\sqrt{2} + 2\pi - 6\cos^{-1}\tfrac{1}{3}) = .20775r^3.$$

[For want of room we are obliged to defer publishing the solution of 29 till next month.]

PROBLEMS.

34. BY PROF. M. L. COMSTOCK, GALESBURG, ILL.—Given $xyz = 18$, (1); $x^2 + y^2 + z^2 = 33$, (2); $(x^2 - yz)^3 + (y^2 - xz)^3 + (z^2 - xy)^3 - 3(x^2 - yz)(y^2 - xz)(z^2 - xy) = 6561$, (3); to find x, y and z.

35. BY CAPT. O. E. MEIBAELIS, PITTSBURGH, PA.—A man let a stone weighing 40 lbs. to a neighbor—the latter broke it accidentally into four parts—and upon returning the fragments consoled the owner by remarking that now he could weigh all numbers between one and forty. In other words, given $a + b + c + d = 40$, to determine such values for a, b, c and d, as will, by *association*, produce all numbers from one to forty.

36. BY HENRY A. ROLAND, TROY, N. Y.—A perfectly flexible cord of given length is suspended from two points whose coordinates are x', y' and x'', y''. How must the weight of the cord vary from point to point in order that it may hang in the arc of a circle.

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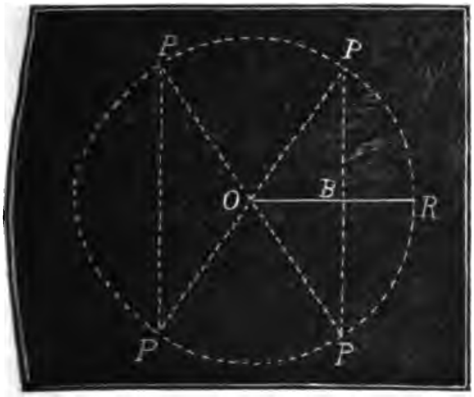
INVOLUTION AND EVOLUTION OF IMAGINARY QUANTITIES CONSIDERED GEOMETRICALLY.

BY PROF. W. D. HENKLE, SALEM, OHIO.

In the diagram let the radius of the circle be 1 and OR its initial position. Conceive the radius to revolve as the hands of a clock. Let OP represent any position of the radius.



Draw PB perpendicular to the line of initial direction. Put OB $=a$ and PB $=b$. Then $a+b\sqrt{-1}$ represents OP, a being positive when P is in the right half of the circumference and negative when P is in the left half of the circumference, and b being positive when P is in the upper half of the circumference and negative when P is in the lower half of the circumference. Let C represent the circumference of the circle, and $\theta=$ RP.

Proposition 1.—The following equations indicate the powers of OP:

$$\begin{array}{lllll}
& OP^0 = OR, & & & \\
& OP^1 = OR & \text{revolved through} & & 1\theta, \\
& OP^2 = OR & `` & `` & 2\theta, \\
& OP^3 = OR & `` & `` & 3\theta, \\
\text{and in general} & OP^n = OR & `` & `` & n\theta.
\end{array}$$

Demonstration.—Since $a=\cos\theta$ and $b=\sin\theta$ we have

$$a + b\sqrt{-1} = \cos\theta + \sqrt{-1}\,\sin.\theta.$$

But we have by De Moivre's formula

$$(\cos\theta + \sqrt{-1}\,\sin\theta)^n = \cos n\theta + \sqrt{-1}\,\sin.n\theta.$$

Hence $\qquad \mathrm{OP}^n = \cos n\theta + \sqrt{-1}\,\sin.n\theta,$

or $\qquad \mathrm{OP}^n = \mathrm{O}R$ revolved through $n\theta$.

Proposition 2.—The following equations indicate the roots of OP:

$\mathrm{OP}^{\frac{1}{1}} = \mathrm{OR}$ revolved through $\frac{1}{1}\theta$,

$\mathrm{OP}^{\frac{1}{2}} = \mathrm{OR}$ " " $\frac{1}{2}\theta$,

$\mathrm{OP}^{\frac{1}{3}} = \mathrm{OR}$ " " $\frac{1}{3}\theta$,

and in general $\mathrm{OP}^{\frac{1}{n}} = \mathrm{OR}$ " " $\frac{1}{n}\theta$.

Demonstration.—Putting $\frac{1}{n}$ for n in De Moivre's formula we have

$$(\cos\theta + \sqrt{-1}\,\sin\theta)^{\frac{1}{n}} = \cos\tfrac{1}{n}\,\theta + \sqrt{-1}\,\sin\tfrac{1}{n}\,\theta.$$

Hence $\qquad \mathrm{OP}^{\frac{1}{n}} = \cos\frac{1}{n}\,\theta + \sqrt{-1}\,\sin\frac{1}{n}\,\theta,$

or $\quad \mathrm{OP}^{\frac{1}{n}} = \mathrm{O}R$ revolved through $\frac{1}{n}\,\theta$.

Revolving OR through $\frac{1}{n}$ of θ gives but one of the n^{th} roots of OP, which I shall call the first of the n roots. The following proposition shows how to get any one of the remaining $n - 1$ roots. Let $m > 1$ and $< n + 1$.

Proposition 3.—The following equations indicate the $n - 1$ of the n roots of OP after the first:

The second of the n roots of $\mathrm{OP} = \mathrm{OP}^{\frac{1}{n}}$ revolved through $\frac{1}{n}$ C,

" third " " n " " $\mathrm{OP} = \mathrm{OP}^{\frac{1}{n}}$ " " $\frac{2}{n}$ C,

" fourth " " n " " $\mathrm{OP} = \mathrm{OP}^{\frac{1}{n}}$ " " $\frac{3}{n}$ C,

" m^{th} " " n " " $\mathrm{OP} = \mathrm{OP}^{\frac{1}{n}}$ " " $\frac{m-1}{n}$ C.

We can make the reference to OR by adding $\frac{1}{n}\,\theta$.

The m^{th} of the n roots of OP $=$ OR revolved through $\frac{1}{n}\,\theta + \frac{1}{n}(m - 1)$C.

Demonstration.—Put OP_m to represent the m^{th} of the n roots of OP, then if the proposition is true $OP_m^n = OP$.

By prop. 1 $OP_m^n = OR$ revolved through $n(\frac{1}{n}\theta + \frac{1}{n}(m-1)C)$,

or $OP_m^n = OR$ " " θ,

since complete circumferences may be rejected.

But $OP = OR$ revolved through θ, therefore $OP_m^n = OP$.

PROPOSITION 4.—$OR^{\frac{1}{n}} = OR$ revolved through $\frac{1}{n}$ C.

Hence when $OP = OR^{\frac{1}{n}}$, OP is the n^{th} root of $+1$, or OR.

Demonstration.—When $\theta =$ RP becomes C, OP becomes OR, hence by substitution $OP^{\frac{1}{n}} = OR$ revolved through $\frac{1}{n}$ C (prop. 2),

becomes $OR^{\frac{1}{n}} = OR$ " " $\frac{1}{n}$ C.

This formula gives but one of the n roots of $+1$, or OR, which root I call the first of the n roots of $+1$.

By prop, 3 the m^{th} of the $n-1$ remaining roots of $+1$, is the m^{th} power of the n^{th} root of $+1$.

It is plain that the n roots of $+1$ are *numerically* equal to each other but *geometrically* or *positionally* different.

The reader may test the above propositions by solving the binomial equations $x^2 = 1$, $x^3 = 1$, $x^4 = 1$, $x^5 = 1$, $x^6 = 1$, $x^8 = 1$, $x^{10} = 1$, $x^{12} = 1$, $x^{20} = 1$, etc., and constructing the roots in accordance with the convention that $\sqrt{-1}$ denotes *perpendicularity*. See my first article in the January ANALYST. The passage from $OR = 1$ to $OR = a$ is simple. The next article will deal with imaginary exponents.

SOLUTION OF AN INDETERMINATE PROBLEM.

BY GEO. R. PERKINS, L L. D., UTICA, N. Y,

No. 5.—*Find three right triangles having equal perimeters, and whose areas shall be in arithmetical progression.*

Since these triangles have the same perimeters, and their areas are in arithmetical progression, it is evident that the radii of their inscribed circles must be in arithmetical progression. We will therefore assume as follows:

$$\text{First triangle.}\quad \begin{cases} r + s & = \text{radius of inscribed circle.} \\ x + (r + s) & = \text{the base.} \\ y + (r + s) & = \text{the perpendicular.} \end{cases}$$

$$\text{Second triangle.}\quad \begin{cases} r & = \text{radius of inscribed circle.} \\ x' + r & = \text{the base.} \\ y' + r & = \text{the perpendicular.} \end{cases}$$

$$\text{Third triangle.}\quad \begin{cases} r - s & = \text{radius of inscribed circle.} \\ x'' + (r - s) & = \text{the base.} \\ y'' + (r - s) & = \text{the perpendicular.} \end{cases}$$

Denoting the common perimeter of the three triangles by p, we have for their hypothenuses as follows:

$$x + y = \tfrac{1}{2}p - (r + s \ldots\ldots\ldots\ldots\ldots\ldots(1),$$
$$x' + y' = \tfrac{1}{2}p - r \ldots\ldots\ldots\ldots\ldots\ldots(2),$$
$$x'' + y'' = \tfrac{1}{2}p - (r - s)\ldots\ldots\ldots\ldots\ldots(3).$$

These conditions give $(x + y) + (x'' + y'') = 2(x' + y')$, which proves the hypothenuses to be in arithmetical progression.

Since the triangles are right, we have

$$[x + (r + s)]^2 + [y + (r + s)]^2 = (x + y)^2 \ldots\ldots\ldots(4),$$
$$[x' + r]^2 + [y' + r]^2 = (x' + y')^2 \ldots\ldots\ldots(5),$$
$$[x'' + (r - s)]^2 + [y'' + (r - s)]^2 = (x'' + y'')^2 \ldots\ldots(6).$$

These, when simplified, become

$$xy = (r + s)(x + y) + (r + s)^2 \ldots\ldots\ldots\ldots(7),$$
$$x'y' = r(x' + y') + r^2 \ldots\ldots\ldots\ldots(8),$$
$$x''y'' = (r - s)(x'' + y'') + (r - s)^2 \ldots\ldots\ldots(9).$$

Using the values of (1), (2) ond (3), we have

$$xy = \tfrac{1}{2}p(r + s)\ldots\ldots\ldots\ldots(10),$$
$$x'y' = \tfrac{1}{2}pr \ldots\ldots\ldots\ldots(11),$$
$$x''y'' = \tfrac{1}{2}p(r - s)\ldots\ldots\ldots(12).$$

If from the squares of (1), (2) and (3) we subtract four times these products, and extract the square roots, we find

$$x - y = \sqrt{\tfrac{1}{4}p^2 - 3p(r + s) + (r + s)^2}\ldots\ldots\ldots\ldots(13),$$
$$x' - y' = \sqrt{\tfrac{1}{4}p^2 - 3pr + r^2} \ldots\ldots\ldots\ldots(14),$$
$$x'' - y'' = \sqrt{\tfrac{1}{4}p^2 - 3p(r - s) + (r - s)^2}\ldots\ldots\ldots\ldots(15).$$

Hence, we must have

$$\tfrac{1}{4}p^2 - 3p(r+s) + (r+s)^2 = \square \ldots\ldots\ldots\ldots (16),$$
$$\tfrac{1}{4}p^2 - 3pr + r^2 = \square \ldots\ldots\ldots\ldots (17),$$
$$\tfrac{1}{4}p^2 - 3p(r-s) + (r-s)^2 = \square \ldots\ldots\ldots\ldots (18).$$

Condition (16) will be satisfied if we take $p = 12(r+s)$, and (17) and (18) will become

$$r^2 + 36rs + 36s^2 = \square \ldots\ldots\ldots\ldots (19),$$
$$r^2 + 70rs + 73s^2 = \square \ldots\ldots\ldots\ldots (20).$$

Conditions (19) and (20) will be satisfied, by taking $r = 9s$, and consequently $p = 120s$.

Taking $s = 1$, we immediately obtain

$$x + y = 50; \quad x' + y' = 51; \quad x'' + y'' = 52;$$
$$x - y = 10; \quad x' - y' = 21; \quad x'' - y'' = 28.$$

Consequently, $x = 30$; $y = 20$;
$x' = 36$; $y' = 15$;
$x'' = 40$; $y'' = 12$.

First triangle.
$$\begin{cases} x + r + s = 40 = \text{base.} \\ y + r + s = 30 = \text{perpendicular.} \\ x + y = \underline{50} = \text{hypothenuse.} \\ 120 = \text{perimeter.} \\ 600 = \text{area.} \end{cases}$$

Second triangle.
$$\begin{cases} x' + r = 45 = \text{base.} \\ y' + r = 24 = \text{perpendicular.} \\ x' + y' = \underline{51} = \text{hypothenuse.} \\ 120 = \text{perimeter.} \\ 540 = \text{area.} \end{cases}$$

Third triangle.
$$\begin{cases} x'' + r - s = 48 = \text{base.} \\ y'' + r - s = 20 = \text{perpendicular.} \\ x'' + y'' = \underline{52} = \text{hypothenuse.} \\ 120 = \text{perimeter.} \\ 480 = \text{area.} \end{cases}$$

I presume these results are the least values which will satisfy our Problem.

REMARK.—This Problem is the same as Problem 103, on page 465

of *John D. Williams' Elementary Treatise on Algebra*, Boston, 1840. The solution there given is satisfactory, but extremely lengthy, and very complex. The results obtained are as follows:

The sides of the first triangle	18601944; 13951458; 23252430.
" " " " second triangle	13999464; 18559223; 23247145.
" " " " third triangle	18515584; 14048388; 23241860.

Common perimeter = 55805832.

Radii of the inscribed circles are 4650486; 4655771; 4661056.

SOLUTION OF A PROBLEM.

BY PROF. DAVID TROWBRIDGE, WATERBURGH, N. Y.

Required the Sum of the Products of n Quantities taken m and m together, no two products having the same Factors.

Let the n quantities be $r_1, r_2, r_3, \ldots . r_n$. Let S_m be the required sum. Also let

$$S_1 = R_1 = r_1 + r_2 + \ldots . + r_n,\ R_2 = r_1^2 + r_2^2 + \ldots . + r_n^2, \ldots .$$
$$R_m = r_1^m + r_2^m + \ldots . + r_n^m \quad (1).$$

Now

$$r_1(r_2 + r_3 + \ldots . + r_n) + r_2(r_1 + r_3 + \ldots . + r_n) + \ldots .$$
$$+ r_n(r_1 + r_2 + \ldots . + r_{n-1}) = r_1(R_1 - r_1) + r_2(R - r_2) + \ldots$$
$$+ r_n(R_1 - r_n) = R_1^2 - R_2 = R_1 S_1 - R_2.$$

But we have evidently taken each product twice, so that we shall consequently have

$$2S_2 = R_1 S_1 - R_2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

$$r_1(r_2 r_3 + r_2 r_4 + \ldots .) + r_2(r_1 r_3 + r_1 r_4 + \ldots .) + \ldots .$$
$$+ r_n(r_1 r_2 + r_1 r_3 + \ldots .) = r_1[S_2 - r_1(R_1 - r_1)]$$
$$+ r_2[S_2 - r_2(R_1 - r_2)] + \ldots . + r_n[S_2 - r_n(R_1 - r_n)]$$
$$= R_1 S_2 - R_1 R_2 + R_3 = R_1 S_2 - R_2 S_1 + R_3.$$

But we have now taken each product three times. We hence have

$$3S_3 = R_1 S_2 - R_2 S_1 + R_3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

$r_1(r_2r_3r_4 + r_2r_3r_5 + \ldots) + r_2(r_1r_3r_4 + r_1r_3r_5 + \ldots) + \ldots$

$= r_1 \left\{ S_3 - r_1[S_2 - r_1(R_1 - r_1)] \right\} + r_2 \left\{ S_3 - r_2[S_2 - r_2(R_1 - r_2)] \right\}$

$+ \ldots \quad = R_1S_3 - R_2S_2 + R_3R_1 - R_4 = R_1S_3 - R_2S_2$
$+ R_3S_1 - R_4 \ldots$

$$\therefore 4S_4 = R_1S_3 - R_2S_2 + R_3S_1 - R_4 \ldots\ldots\ldots\ldots\ldots\ldots (4.)$$

We now easily see that the general equation will be

$$mS_m = R_1S_{m-1} - R_2S_{m-2} + R_3S_{m-3} - \ldots + (-1)^{m-2}R_{m-1}S_1 + (-1)^{m-1}R_m \ldots\ldots\ldots\ldots (5.)$$

The general equation of the n^{th} degree is

$$x^n + A_1x^{n-1} + A_2x^{n-2} + \ldots + A_n \ldots\ldots\ldots\ldots\ldots\ldots (6.)$$

Let the n roots of this equation be $r_1, r_2, \ldots r_n$, so that (6) becomes (not regarding the signs of the roots)

$$(x + r_1)(x + r_2)(x + r_3)\ldots(x + r_n) = x^n + S_1x^{n-1} + S_2x^{n-2} + \ldots (7).$$

That is

$$S_1 = A_1,\ S_2 = A_2,\ S_3 = A_3,\ \&c.$$

By means of equation (2) we have

$$R_2 = S_1R_1 - 2S_2 = S_1^2 - 2S_2 = A_1^2 - 2A_2 \ldots\ldots\ldots\ldots (8).$$

From (3)

$$R_3 = 3S_3 - R_1S_2 + R_2S_1 = 3A_3 - A_1A_2 + A_1(A_1^2 - 2A_2)$$
$$= 3A_3 - 3A_1A_2 + A_1^3 \ldots\ldots\ldots\ldots\ldots\ldots (9).$$

In a similar manner we find

$$R_4 = A_1^4 - 4A_1^2A_2 + 2A_2^2 + 4A_1A_3 - 4A_4 \ldots\ldots\ldots\ldots\ldots (10).$$

It is also evident that we could easily deduce the values of R_5, R_6, &c. We have therefore found, in this way, the sum of the squares, the sum of the cubes, the 4th powers, and so on, of the roots of an equation in terms of the coefficients of the equation. These powers may extend up to the n^{th}, but not beyond, since $S_n = r_1r_2r_3 \ldots\ldots r_n$, and we cannot have a product with any more factors than n.

From (5) we see that

$$r_1 r_2 r_3 \ldots r_n = \tfrac{1}{n}[R_1 S_{n-1} - R_2 S_{n-2} + \ldots + (-1)^{n-2} R_{n-1} S_1 + (-1)^{n-1} R_n].$$

If $n = 2, 3, 4$, &c.,

$$r_1 r_2 = \tfrac{1}{2}[(r_1 + r_2)^2 - r_1^2 - r_2^2];\ r_1 r_2 r_3 = \tfrac{1}{3}[\tfrac{1}{2}(r_1 + r_2 + r_3)^3 - \tfrac{1}{2}(r_1 + r_2 + r_3)(r_1^2 + r_2^2 + r_3^2) - (r_1 + r_2 + r_3)(r_1^2 + r_2^2 + r_3^2) + r_1^3 + r_2^3 + r_3^3] = \tfrac{1}{3}[\tfrac{1}{2}(r_1 + r_2 + r_3)^3 - \tfrac{3}{2}(r_1 + r_2 + r_3)(r_1^2 + r_2^2 + {}^2_3) + r_1^3 + r_2^3 + r_3^3].$$

These transformations show how easily we may derive results by a general process, that would otherwise be difficult.

DEMONSTRATION OF TWO OF PROF. PEIRCE'S PROPOSITIONS.

BY PROF. M. L. COMSTOCK, OF KNOX COLLEGE, ILL.

In the *Mathematical Montlhty* (Runkle's), No. 1, Vol. 1, Prof. Peirce gave a set of "Propositions on the Distribution of Points on a Line." They were subsequently solved by Prof. W. P. G. Bartlett and R. J. Adcock. I send you a solution of two of them by a method presumably applicable to the others.

Prop. 1.—"If two points are taken in all possible relative positions, upon a given line, their distance apart in one-half the whole number of possible positions, is less than .29289 of the length of the line."

Prop. 2.—"The number of positions in which the distance between the points is less than half the length of the line is $\frac{3}{4}$ of the whole number of possible positions."

If a = the length of the line, and x = the shortest distance between two points, it readily appears that

$\frac{a}{x} =$	number of positions at the distance			x,
$\frac{a}{x} - 1 =$	"	"	"	$2x$.
$\frac{a}{x} - 2 =$	"	"	"	$3x$,
$\vdots$				
$2 =$	"	"	"	$a-x$,
$1 =$	"	"	"	a,

Hence $\frac{a}{2x}\left(\frac{a}{x} + 1\right)$ = whole number of possible positions, since these quantities are in arithmetical progression, and the number of terms is $a \div x$.

Notice also that, if x is taken as the measuring unit, the number of the term is the same as the distance between the points in that term.

Now let z = the number of terms which shall embrace half of an arithmetical reries whose first term is t, last term 1, com. dif. 1, and the number of terms n.

The sum embraced by z will be

$$\tfrac{1}{2}z(2t - z + 1);$$

also, $t - z$ is the first term of what remains of the given series, 1 is the last term, and $n - z$ the number of terms; hence the sum is

$$\tfrac{1}{2}(n - z)(t - z + 1),$$

$$\tfrac{1}{2}z(2t - z + 1) = \tfrac{1}{2}(n - z)(t - z + 1),$$

$$\text{and } z^2 - \tfrac{1}{2}(3t + 2 + n)\,z = -\tfrac{1}{2}(z + 1)\,n.$$

Now in our example,

$a \div x$ = number of positions at the distance $x = t$,

$a \div x = n$, as readily appears in our series, hence,

$$z^2 - \left(2\frac{a}{x} + 1\right)z = -\frac{1}{2}\left(\frac{a^2}{x^2} + \frac{a}{x}\right);$$

and making x infinitely small,

$$z^2 - \frac{2a}{x}z = -\frac{1}{2}\frac{a^2}{x^2},\ z = \left(1 - \frac{\sqrt{2}}{2}\right)\frac{a}{x} = (.29289-)\frac{a}{x},$$

which establishes Prop. 1, x being called the measuring unit.

Again, if we place the points at a distance from each other equal to half the line we shall have

$\frac{a}{2x} + 1$ = number of positions at the distance $\frac{a}{2}$,

$\frac{a}{2x}$ = " " " " $\frac{a}{2} + x$,

$\frac{a}{2x} - 1$ = " " " " $\frac{a}{2} + 2x$,

⋮

2 = " " " " $a - x$,

1 = " " " " a.

Here the number of terms is $\frac{a}{2x}+1$, and the sum

$=\frac{1}{2}\left(\frac{a}{2x}+1\right)\left(\frac{a}{2x}+2\right)=$ number of positions in which the distance between the points is *not less* than half the line.

But the whole number of positions as seen before is $\frac{a}{2x^2}\ (a+x)$.

When x is taken very small, these quantities reduce to $\frac{a^2}{8x^2}$ and $\frac{a^2}{2x^2}$, the first part of which is $\frac{1}{4}$ the second, hence the whole number of positions in which the greatest distance is *less* than half the line, is $\frac{3}{4}$ of the whole number of possible positions.

SOLUTIONS OF PROBLEMS IN NO. 7, *AND* 29, *IN NO.* 6.

Solutions of problems in No. 7 have been received as follows:

From Geo. L. Dake, 30, 31 & 32; Geo. M. Day, 30, 32 & 33; Prof. A. B. Evans, 30, 31, 32 & 33; Prof. H. T. Eddy, 32; Henry Gunder, 30, 31, 32 & 33; H. Heaton, 30, 31, 32 & 33; Prof. E. W. Hyde, 32 & 33; Prof. D. Kirkwood, 31; Artemas Martin, 30, 31, 32 & 33; L. Regan, 30 & 31; Walter Siverly, 30, 31, 32 & 33; Werner Stille, 30, 31, 32 & 33; E. B. Seitz, 30. 31, 32 & 33; Prof. D. Trowbridge, 30, 31 & 32.

Elegant solutions of all the questions proposed in No. 6 were received in due time, last month, from Walter Siverly, but the letter containing his solutions of 25, 26 and 29 was mis-laid and hence proper credit was not given him in No. 8. Mr. Martin also sent correct solutions of Nos. 25 and 26, but failed to get credit for them in No. 8, because his letter, also, was mis-laid.

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29. "If $a, b, c, d, e, f, g, h, i, j, k$ be chords drawn from any point on the circumference of a circle to the eleven angles of an inscribed regular polygon of eleven sides; prove that

$$(a+k)(b+j)(c+i)(d+h)(e+g)=f^5 \quad\ldots\ldots\ldots\ldots\ldots\ldots (1)."$$

GENERAL SOLUTION, BY PROF. A. HALL.

Put the radius of the circle equal to unity, and let the number of the sides of the polygon be n, an odd number. Then $\cos\theta$ and $\sin\theta$ are the coordinates of a point on the circumference, and

$$(1,0);\left(\cos\frac{2\pi}{n},\ \sin\frac{2\pi}{n}\right);\quad\left(\cos\frac{4\pi}{n},\ \sin\frac{4\pi}{n}\right)\ldots\left(\cos\frac{(n-1)\pi}{n},\ \sin\frac{(n-1)\pi}{n}\right),$$

are the coordinates of the corners of the polygon. Denote the chords by c_1; c_2; c_3;........c_n. We shall have by the common formulae of analytical geometry, or from the geometry of the figure,

$$c_1=2\sin\frac{1}{2}\theta,\qquad c_2=2\sin\left(\frac{\theta}{2}-\frac{\pi}{n}\right),$$

$$c_3=2\sin\left(\frac{\theta}{2}-\frac{2\pi}{n}\right),\qquad c_4=2\sin\left(\frac{\theta}{2}-\frac{3\pi}{n}\right),$$

$$\ldots\ldots\ldots\ldots\qquad\ldots\ldots\ldots\ldots$$

$$c_{n-1}=2\sin\left(\frac{\theta}{2}-\frac{(n-2)\pi}{n}\right),\qquad c_n=2\sin\left(\frac{\theta}{2}-\frac{(n-1)\pi}{n}\right).$$

Adding these values in the manner indicated we have

$$c_1+c_n=4\sin\left(\frac{\theta}{2}-\frac{(n-1)\pi}{2n}\right)\cos\frac{(n-1)\pi}{2n},$$

$$c_2+c_{n-1}=4\sin\left(\frac{\theta}{2}-\frac{(n-1)\pi}{2n}\right)\cos\frac{(n-3)\pi}{2n},$$

$$c_3+c_{n-2}=4\sin\left(\frac{\theta}{2}-\frac{(n-1)\pi}{2n}\right)\cos\frac{(n-5)\pi}{2n},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$c_{\frac{n-1}{2}}+c_{\frac{n+3}{2}}=4\sin\left(\frac{\theta}{2}-\frac{(n-1)\pi}{2n}\right)\cos\frac{2\pi}{2n},$$

and for the middle chord,

$$c_{\frac{n+1}{2}}=2\sin\left(\frac{\theta}{2}-\frac{(n-1)\pi}{2n}\right).$$

The product of the sums is therefore

$$2^{n-1}.\sin\left(\frac{\theta}{2}-\frac{(n-1)\pi}{2n}\right)^{\frac{n-1}{2}}.\cos\frac{(n-1)\pi}{2n}.\cos\frac{(n-3)\pi}{2n}.\cos\frac{(n-5)\pi}{2n}\ldots\cos\frac{2\pi}{2n}.$$

Now from a known theorem we can deduce easily the equation,

$$2^{\frac{n-1}{2}} \cdot \cos\frac{(n-1)\pi}{2n} \cdot \cos\frac{(n-3)\pi}{2n} \cdot \cos\frac{(n-5)\pi}{2n} \cdots\cdots \cos\frac{2\pi}{2n} = 1,$$

and therefore we have

$$c^{\frac{n-1}{2}}_{\frac{n+1}{2}} = (c_1 + c_n)(c_2 + c_{n-1})(c_3 + c_{n-2})(c_4 + c_{n-3}) \ldots (c_{\frac{n-1}{2}} + c_{\frac{n+3}{2}}).$$

---

30. "Prove that

$$\frac{\sqrt[4]{a} + \sqrt[4]{b}}{\sqrt[4]{a} - \sqrt[4]{b}} = \frac{a + b + 2\sqrt{ab} + 2\sqrt[4]{a^3 b} + 2\sqrt[4]{ab^3}}{a - b}."$$

SOLUTION BY GEO. L. DAKE, CLEVELAND, OHIO.

Multiply both numerator and denominator of the given fraction by $(\sqrt{a} + \sqrt{b})(\sqrt[4]{a} + \sqrt[4]{b})$, reduce, and we shall have

$$\frac{a + b + 2\sqrt{ab} + 2\sqrt[4]{a^3 b} + 2\sqrt[4]{ab^3}}{a - b}.$$

---

31. "Solve the equation

$$x = \sqrt[4]{x + \frac{x}{\sqrt[4]{x + \frac{x}{\sqrt[4]{x + \&c}}}}}$$

and express the value of x in a finite number of terms."

SOLUTION BY ASHER B. EVANS, LOCKPORT, N. Y.

By the theory of continued fractions the given equation becomes

$$x = \sqrt[4]{x + x \div x} = \sqrt[4]{x + 1} \quad \therefore x^4 = x + 1 \ldots\ldots\ldots\ldots\ldots (1).$$

Assume $$(x^2 + p)^2 = (qx + r)^2 \ldots\ldots\ldots\ldots\ldots (2).$$

Equations (1) and (2) will be identical when $2p = q^2$, $2qr = 1$, and $r^2 - p^2 = 1$. By eliminating q and r these three conditions give

$$8p^3 + 8p - 1 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Put $2p = y - \frac{4}{3y}$; then (3) becomes $y^6 - y^3 = \frac{64}{27}$; whence

$y = \left(\frac{1}{2} + \frac{1}{2}\sqrt{\frac{283}{27}}\right)^{\frac{1}{3}}$; and therefore

$$p = \frac{1}{2}y - \frac{2}{3y} = \frac{1}{2}\left(\frac{1}{2} + \frac{1}{2}\sqrt{\frac{283}{27}}\right)^{\frac{1}{3}} + \frac{1}{2}\left(\frac{1}{2} - \frac{1}{2}\sqrt{\frac{283}{27}}\right)^{\frac{1}{3}} \ldots\ldots(4).$$

Since $q = \sqrt{2p}$ and $r = \frac{1}{2q} = \frac{1}{2\sqrt{2p}}$, we find from (2) by substituting these values of q and r

$$(x^2 - p)^2 = \left(x\sqrt{2p} + \frac{1}{2\sqrt{2p}}\right)^2.$$

$$\therefore x^2 - x\sqrt{2p} = \frac{1}{2\sqrt{2p}} - p \text{ and}$$

$$x^2 + x\sqrt{2p} = -\frac{1}{2\sqrt{2p}} - p; \text{ whence}$$

$$x = \sqrt{\frac{1}{2}}\left(\sqrt{p} \pm \sqrt{\frac{1}{\sqrt{2p}} - p}\right) \text{ and}$$

$$x = \sqrt{\frac{1}{2}}\left(-\sqrt{p} \pm \sqrt{-\frac{1}{\sqrt{2p}} - p}\right.$$

---

32. "There are two spheres of equal size and of exactly the same appearance, one solid silver and galvanized with gold and the other hollow and made of gold. Required some means by which one may be determined from the other."

SOLUTION BY PROF. D. TROWBRIDGE, WATERBURGH, N. Y.

Convert the spheres into pendulums, and let a be the distance from the point of suspension to the centre of the spheres, and l the distance from the same point to the centre of oscillation in the solid sphere and l' in the hollow one, k the principal radius of gyration for the solid sphere, and k' for the other. Then

$$l = \frac{k^2 + a^2}{a}, \qquad l' = \frac{k'^2 + a^2}{a}.$$

If r and r' be the radii of the external and the internal surface of the hollow sphere, then

$$k^2 = \frac{2}{5}\,r^2, \text{ and } k'^2 = \frac{2}{5}\cdot\frac{r^5 - r'^5}{r^3 - r'^3} = \frac{2}{5}\left(r^2 + r'^2) - \frac{r^2 r'^2}{r^2 + rr' + r'^2}\right).$$

$\therefore\ k' > k$, and $t' > t$; or the solid sphere will vibrate in the less time.

[H. T. Eddy, Prof. of Math. and Astronomy, Cincinnati, Ohio, has favored us with an elaborate solution of this question. Prof. Eddy supposes the spheres to be rolled down an inclined plane and shows that the solid ball will roll more rapidly than the other and reach the bottom more quickly. He obtains the eq., $t' = \sqrt{(k'^2 + r^2) \div (k^2 + r^2)}\,t$, in which t is the time and k the radius of gyration of the silver sphere, and t' the time and k' the radius of gyration of the golden sphere; and, assuming the spheres to be of equal weight, he finds $k^2 = \frac{2}{5}r^2$ and $k'^2 = \frac{2}{5}\cdot\frac{781}{593}r^2$, and hence $t' = 1.044t$, nearly. We intended to publish this solution in full, but find it too extended for our space.—Ed.]

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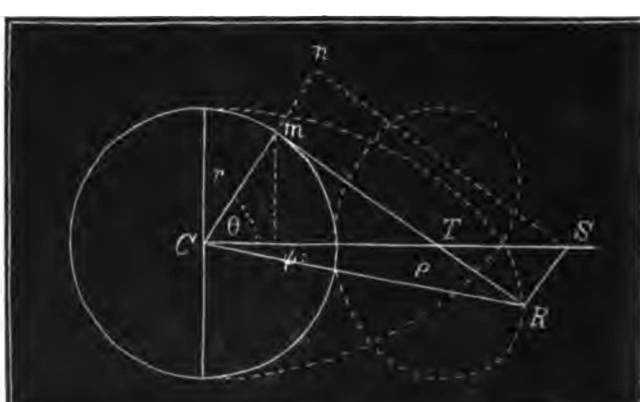
33. "A circle is referred to rectangular axes passing through the center. A tangent and ordinate are drawn from any point of the circumference. A distance n times the abscissa of this point is measured upon the axis of x and from this point a perpendicular falls upon the tangent. Required the equation of the locus of the intersection of the perpendicular and tangent."

SOLUTION BY WERNER STILLE, MARINE, ILL.



Let MTR be the tangent at M; CS $= nx$; MCS$=\theta$; SCR $= \psi$; then from the triangle CMR, when CM $= r$, and CR $= \rho$,

(1)............$\rho.\cos(\psi + \theta) = r$.

Also, when NS = MR, because $x = r\cos\theta$, CS $= n.r.\cos\theta$.

(2)............$\rho.\sin(\psi + \theta) = n.r.\cos\theta.\sin\theta$.

(2) can be written in the form, $\rho.\sin(\psi + \theta) = \frac{1}{2}\, n.r.\sin 2\theta$.

From this we easily deduce

$$\cos(\psi + \theta) = \sqrt{1 - \frac{n^2r^2}{4\rho^2}.\sin^2 2\theta},$$

which, when inserted in (1), gives

$$\rho = \frac{r}{\sqrt{1 - \frac{n^2r^2}{4\rho^2}.\sin^2 2\theta}}.$$

Finally, resolving to a function of ρ, we have,

(3)............$\rho = \pm r\sqrt{1 + \frac{n^2}{4}\sin^2 2\theta}$, the equation required.

The double sign shows that the curve consists of two equal branches in symmetrical position. The figure represents the curve, approximately, when $n = 4$.

[This solution gives the equations between the radius-vector and the *angle* subtended by the axis of x and the perpendicular upon the tangent. From it, however, we may readily obtain the equation between the radius-vector and the perpendicular (P) upon the tangent. We have $P = n\cos^2\theta - 1 = n(1 - \sin^2\theta) - 1 = n - 1 - n\sin^2\theta$; and from (3) we have $\sin\theta = \pm\sqrt{\frac{1}{2} \pm \frac{1}{2n}\sqrt{n^2 - \rho^2 + 4}}$.

$$\therefore\ P = \frac{n-2}{2} \mp \frac{1}{2}\sqrt{n^2 - 4\rho^2 + 4}\ \ldots\ldots(4).$$

Or, because $\rho : P :: \sin\theta : \sin\psi$ we have $\rho\sin\psi = P\sin\theta$, or, by substitution,

$$\rho\sin\psi = \left(\frac{n-2}{2} \mp \frac{1}{2}\sqrt{n^2 - 4\rho^2 + 4}\right)\sqrt{\frac{1}{2} \pm \frac{1}{2n}\sqrt{n^2 - \rho^2 + 4}}\ \ldots\ldots(5),$$

which corresponds with the eq. obtained by Prof. Evans by a different process of reasoning.—Ed.]

Dr. Jas. Matteson, of De Kalb Center, Ill., has sent us an elaborate solution of No. 5, which we are sorry our space will not permit us to publish.

Putting x, y and z for the three sides of the triangle, and 5, 7 and 9 for the three bisecting lines, and, moreover, putting $mx = y$ and $nx = z$, the Dr. gets, $m = .5814480988564$, and $n = 799178525271$5.

$$x = \frac{5}{\sqrt{mn - \frac{mn}{m+n}}},\ y = \frac{7}{\sqrt{m - \frac{mn^2}{(m+1)^2}}} \text{ and } z = \frac{9}{\sqrt{n - \frac{m^2n}{(n+1)^2}}}.$$

PROBLEMS.

37. By G. Shaw, Kemble, Ontario, Canada.—Divide unity into three parts such that if each part be increased by unity the sums shall be three rational cubes.

38. By Asher B. Evans, A. M., Lockport, N. Y.—Prove that when x is infinite

$$\frac{1}{\lfloor 2} - \frac{1}{\lfloor 3} + \frac{1}{\lfloor 4} - \ldots\ldots\ldots\ldots \frac{(-1)^x}{\lfloor x} = \frac{1}{e};$$

where e is the base of Napierian logarithms.

39. By Prof. C. M. Woodward, St. Louis, Mo.—Two weights are connected by a fine string which passes over a pulley; if the weights be 50 and 72 lbs., determine what stationary weight the string must be able to support, that it may just escape breaking during the motion.

40. By the Editor.—P and Q are two points, distant apart a miles on the bank of a straight canal, through which there flows, from Q towards P, a uniform current with the velocity of m miles per hour. R is a point on the opposite bank of the canal, at right angles from P with the line PQ. Two men, A and B, start at the same moment from the two points P and R; A starts from P and *walks* directly toward Q with the velocity of n miles per hour, and B starts from R, with a boat, and *rows*, with a constant effort that in the absence of a current, would carry him r miles per hour, and endeavors to join A by rowing continually directly towards him, and succeeds in joining A at the moment he arrives at Q. What is the width of the canal? or, distance between P and R?

BOOK NOTICES.

Books and pamphlets have been received as follows:

Notes on Mechanics, designed to be used in connection with Rankin's Applied Mechanics, by Gaetano Lanza, S. B., C. E., Assistant Prof. of Math. and Mechanics, Mass. Institute of Technology.

Part 1, Statics, Boston, 1872; 98 p. 12 mo.

" 2, Dynamics, " 1874; 72 p. 12 mo.

This work is not *published*, but merely printed for the use of the students at the Institute of Technology.

The First Differential Coefficient, by John Newton Lyle, A. M., Prof. of Applied Science in Westminster College, Fulton, Mo. St. Louis, Mo., 35 pages 12 mo.

Gravitation to the Sphere and the two Ellipsoids of Revolution and Ratio of the Axes of a Revolving Fluid Mass, by R. J. Adcock, Cincinnati, Ohio, 1872, 11 pages 4to.

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Editorial Note.—Circumstances have made it necessary to issue the present number earlier than usual; and it is probable that we will issue Nos. 11 and 12, consisting of 32 pages, under one cover, early in October, at which time we will make known our designs for the future.

THE ANALYST.

Vol. I. October, 1874. No. 10.

ELEMENTARY TREATMENT OF THE PROBLEM OF TWO BODIES.

BY G. W. HILL.

The deduction of the motion of the planets, in accordance with the laws of Kepler, from the principle of universal gravitation, is important, not only on account of the extensive role this theory plays in Astronomy, but also for its interest, in a historical point of view, as Newton's principal discovery. Hence it is desirable that the demonstration be made as elementary and as brief as possible, in order that it may be brought within the comprehension of the largest number of persons.

The polar equation of the conic section, referred to a focus as pole,

$$r = \frac{a(1 - e^2)}{1 + e\cos(\lambda - \omega)},$$

is well known; a denotes half the greater axis, e the eccentricity and ω the angle made by the axis with the line from which λ is measured. It will be advantageous to replace $a(1 - e^2)$ by p, p being the parameter, also to put

$$\alpha = e\cos\omega, \qquad \beta = e\sin\omega.$$

Thus the equation becomes

$$r + \alpha r\cos\lambda + \beta r\sin\lambda = p.$$

Hence it is plain that the equation, in terms of rectangular coordinates, the origin being at a focus, but the axes of coordinates having any direction we please, is

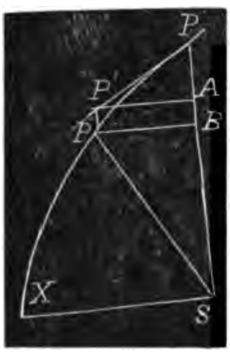
$$\sqrt{x^2 + y^2} + \alpha x + \beta y = p \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

We will take it for granted that the reader is acquainted with the following theorems, since they are demonstrated in all books on mechanics, even the most elementary:

In determining the relative motion of one body about another, it suffices to regard the latter as fixed, and to attribute to it a mass equal to the sum of the masses, and then to suppose the moving body without mass.

When a body describes a plane curve, and the radius vector, drawn from a fixed point in the plane of the curve, passes over equal areas in equal times, (which we shall express by saying that the areolar velocity about the fixed point is constant), the force acts always in the direction of the radius: and the converse.

Let now a body describe a conic section about another occupying a focus, the areolar velocity about this focus being constant: it is required to determine the force acting.



In the figure, let PP″X be an arc of the conic section so described, S being the focus. Let P and P″ be any two points on the curve at an indeterminate but small distance from each other. Draw SP, PP′ a tangent at P, P″P′ parallel to, and P′A and P″B perpendicular to SP. Let SP be taken as the axis of y, and SX perpendicular to it, as the axis of x. The coordinates of P are then, $x = 0, y = SP = r$; substituting these in the equation of the cure, we get

$$(1 + \beta)r = p \dots\dots\dots\dots\dots\dots (2).$$

Since the coordinate x is here supposed always very small, the term $\sqrt{x^2 + y^2}$ in (1) can be expanded, by the binomial theorem, in a series of ascending powers of x. Neglecting x^4 and higher powers we get

$$y + \frac{x^2}{2y} + \alpha x + \beta y = p,$$

or as y differs from r only by a quantity of the order of x, by neglecting x^3,

$$y + \frac{x^2}{2r} + \alpha x + \beta y = p, \qquad y = \frac{p - \alpha x - \dfrac{x^2}{2r}}{1 + \beta}.$$

Or by (2)

$$y = r - \frac{a}{1+\beta}x - \frac{x^2}{2p}.$$

This is the value of y from (1), expanded in a series of ascending powers of x, the cubes and higher powers being omitted. The equation

$$y = r - \frac{a}{1+\beta}x$$

belongs to a right line, which can be nothing else than the tangent PP′. Hence it is plain, from the figure, that taking P″B = P′A = x,

$$\text{tan. PP'A} = \frac{a}{1+\beta} \quad \ldots\ldots\ldots\ldots\ldots\ldots(3),$$

$$\text{PA} = \frac{a}{1+\beta}x,$$

$$\text{P'P''} = \text{AB} = \frac{x^2}{2p} \quad \ldots\ldots\ldots\ldots\ldots\ldots(4),$$

the last equation being only approximate, but more and more nearly true as P″B or x becomes smaller.

Let F denote the force acting on the moving body, and t the small interval of time in which it passes from P to P″. Then we have

$$\text{P'P''} = \frac{x^2}{2p} = \tfrac{1}{2}\text{F}t^2.$$

If we denote double the areolar velocity by h, since P″B $= x$ is very small, we have

$$\text{SP.P''B} = rx = ht.$$

Eliminating t from these equations, we get

$$\text{F} = \frac{h^2}{pr^2}.$$

Since there is no limit to the supposed smallness of x and t, this equation is rigorously exact. The force is then inversely as the square of the radius-vector, and its intensity at the unit of distance is found simply by dividing the square of double the areolar velocity by the semi-parameter. It is evidently attractive, except when, the motion being in a hyperbola, the focus, about which the areolar velocity is constant, is the exterior, in which case it is repulsive.

Taking up the inverse problem, let a body start from P towards P′ with a velocity v which would carry it to the latter point in the time t, and let it be subjected to the action of a force varying inversely as the square of its distance from a second body supposed fixed at S: it is required to fiud the curve described.

Let the masses of the bodies. measured by the velocities they are able to communicate, by their action, in the unit of time and at the unit of distance, be denoted severally by m and M. The force acting at P is then

$$\frac{M+m}{SP^2} = \frac{M+m}{r^2},$$

and, if at the end of the time t, the body is at P″ instead ot P′, we must have

$$P'P'' = \tfrac{1}{2}\frac{M+m}{r^2}t^2.$$

But, as before, the constancy of the areolar velocity gives

$$rx = ht.$$

Whence

$$P'P'' = \frac{M+m}{2h^2}x^2.$$

This equation coincides with (4), if we suppose

$$p = \frac{h^2}{M+m} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (5).$$

Let now a conic section, having this value for its semi-parameter, be described with S as focus and touching PP′ at P. That this is possible, is evident from the general equation (1); here are only two unknowns, α and β, to be determined, and they are given by equations (2) and (3), whence we see the solution is always unique. A body, moving upon this conic section, would have, at the point P, the same velocity, and the same direction of motion, and be subjected to the action of an equa force, and having the same law of variation, as the moving body in the problem. Hence, if the path of the latter is thoroughly determinate, and it would be absurd to suppose otherwise, the conic section just described must be the curve sought.

We can easily find the elements of this conic section. Thus let the angle P′PS be denoted by ψ, then evidently

$$h = rv\sin\psi,$$

which, substituted in (5), gives the value of $p = a(1 - e^2)$; next α and β, which we recall stand for $e\cos\omega$ and $e\sin\omega$, are given by (2) and (3). That is

$$a(1 - e^2) = \frac{r^2v^2\sin^2\psi}{M + m},$$

$$e\cos\omega = \frac{rv^2\sin\psi\cos\psi}{M + m},$$

$$e\sin\omega = \frac{rv^2\sin^2\psi}{M + m} - 1,$$

whence we derive

$$e^2 = 1 - 2\frac{rv^2\sin^2\psi}{M + m} + \frac{r^2v^4\sin^2\psi}{(M + m)^2}, \qquad \frac{1}{a} = \frac{2}{r} - \frac{v^2}{M + m},$$

consequently the greater axis, and the species of conic section described, are independent of ψ. We have an ellipse, a parabola, or a hyperbola, according as v^2 is less, equal to, or greater than $2\dfrac{M + m}{r}$.

From the last equation $v^2 = (M + m)\left(\dfrac{2}{r} - \dfrac{1}{a}\right)$..............(6),

which may evidently be taken as a general expression for the square of the velocity, if r denote the general radius-vector.

Also from (5), $h = \sqrt{(M + m)p}$.

Thus, in different orbits, the areolar velocities are as the square-roots of the parameters, and as the square-roots of the sums of the masses. In an elliptic orbit, if T denote the time of revolution, the double of the area of the whole ellipse

$$hT = 2\pi a^2\sqrt{1 - e^2} = 2\pi a^{\frac{3}{2}}\sqrt{p},$$

whence

$$T = \frac{2\pi a^{\frac{3}{2}}}{\sqrt{M + m}}.$$

Thus the theorem that, provided the sum of the masses remain the

same, the squares of the periodic times in different orbits are as the cubes of the greater axes.

The mean angular velocity is usually denoted by n, thus,

$$n = \frac{2\pi}{T} = \sqrt{\frac{M + m}{a^3}}.$$

It is customary with astronomers to assume the earth's mean distance from the sun as the linear unit. If M and m are the masses severally of the sun and earth, and m', a' and n' belonging to another planet are introduced, the mean distance of the last is given by the equation

$$a' = \left[\frac{1 + \frac{m'}{M}}{1 + \frac{m}{M}} \frac{n^2}{n'^2} \right]^{\frac{1}{3}}.$$

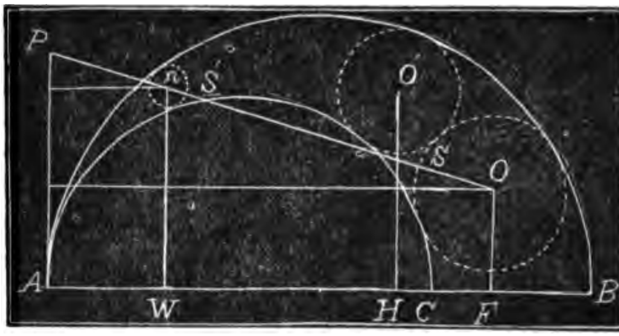
To complete the subject it is necessary to notice a particular case of the problem, viz., when $\psi = 0$. Here the motion is in a right line, and from (6) it appears the velocity is infinite when the body arrives at S. As the existence of another body here ought not to be considered, at least in a mathematical sense, as an obstacle to its further motion, it is plain the body will pass beyond and move in the same right line until its velocity is reduced to zero, when it will return on its path, which will thus be a portion of a right line of which S is the middle point. This cannot be considered as a degenerate form of a conic section of which S is the focus. For when an ellipse is varied by augmenting the eccentricity but maintaining the greater axis constant, at the point the first has attained the limit unity, the ellipse has degenerated into two equal portions of right lines overlapping each other and having their extremities on one side in the point S. Hence this case must be regarded as a singular solution. However most of the properties of motion can be deduced from those of elliptic motion. Thus, if the length of the whole path be denoted by $4a$, the duration of an oscillation will be

$$\frac{2\pi a^{\frac{3}{2}}}{\sqrt{M + m}}.$$

Whence we gather that the time, in which a planet, at rest at its mean distance, would fall to the sun, is found by dividing its periodic time by $4\sqrt{2}$.

GEOMETRICAL DEMONSTRATION OF A THEOREM.

BY ISAAC H. TURRELL, CUMMINSVILLE, OHIO.



Let T be the common tangent, and d_1, d_3 the diameters of the two circles that can be described touching three circles that touch each other, then $T^2 = 4d_1d_3$.

This theorem is given by Matthew Collins, on page 278 of *Mathematical Monthly*, Vol. 1, who states that he proposed it in an old number of the "Educational Times," but no geometrical demonstation had yet appeared. He then gives a solution of a particular case, only, namely, that in which one of the three circles that thouch each other becomes infinite.

The following method pre-supposes some knowledge of the properties of Centers of Similitude, of radical axes and of circles in contact, and it is based on a beautiful theorem, remarkable for its generality, known to the ancients under the name of "The Arbelos," or "The Shoemaker's Knife," which is enunciated by Mr. Collins, in the paper referred to, by means of a figure, as follows:

"If the semicircles on the diameters AB, AC, touching each other at A, be both touched by the circles whose centers are O and O′. which touch each other at L; demit OF, O′H perpendiculars on AB, then if OF $= n$ times the diameter of O, O′H $= (n + 1)$ times the diameter of O′, O′ being nearer io A than O is."

The elegant geometrical proof there given, holds true, if the two original semicircles touched each other externally at A.

Let the circle whose center is O″, touch the circle O′ and the two original semicircles, O‴ touch O″ and the semicircles, and so on in the same order, N being the center of the n^{th} circle from O; it is required to find the relation connecting the quantities T, R and r, T being the

common tangent, and R, r, the radii of the circles whose centers are O and N.

This may be easily accomplished by means of the Arbelos, thus:

Putting OF $=m$ times 2R, then NW $=(m+n).2r$.

$$\frac{NW}{2r}-\frac{OF}{2R}=n. \qquad NW.R - OF.r = 2nRr \ldots\ldots\ldots\ldots(1).$$

Putting ON $=d$, $$d^2 = T^2 + (R-r)^2 \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Now produce ON until it intersects the common tangent, or radical axis of the original semicircles in P; hence by a well-kdown theorem, namely, "If each of two circles touch (in the same way) another pair of circles, the center of similitude of either pair lies upon the radical axis of the other pair," P is the external center of similitude of the circles O and N, and

$$\frac{PO}{PN}=\frac{R}{r} \ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Also S, S′, being two anti-homologous points, (see Chauvenet's, p. 360, or any treatise on the Modern Geometry) on the circles O and N, we have the equation

$$\sqrt{(PN+NS')(PO-OS)}=PA \ldots\ldots\ldots\ldots\ldots\ldots(4).$$

From (3) we get $\dfrac{dr}{R-r}=PN,\ \dfrac{dR}{R-r}=PO.$

Hence (4) becomes

$$\sqrt{\left(\frac{dr}{R-r}+r\right)\left(\frac{dR}{R-r}-R\right)}=\sqrt{\frac{d^2Rr}{(R-r)^2}-Rr}=PA \ldots\ldots\ldots(5).$$

By similar triangles

$$\frac{PA-OF}{PA-NW}=\frac{PO}{PN}=\frac{R}{r}.$$

$PA(R-r)=NW.R-OF.r=2nRr$ from (1).

$PA=\dfrac{2nRr}{R-r}$. By substituting in eq. (5), this value of PA, and the value of d^2 as given in (2), we get

$$\left\{\frac{[T^2+(R-r)^2]Rr}{(R-r)^2}-Rr\right\}^{\frac{1}{2}}=\frac{2nRr}{R-r},$$

which reduces to $T = 2n\sqrt{Rr}$, the relation required; or putting $R = \frac{1}{2}d_1$, $r = \frac{1}{2}d_n$, we get $T = n\sqrt{d_1 d_n}$.

If $n = 2$, we have the original theorem proposed by Mr. Collins.

The same method will apply, if the two original semicircles touched each other externally at A.

If we consider T the transverse, instead of the direct common tangent of O and N, then (2) becomes

$$d^2 = T_1^2 + (R + r)^2.$$

Substitute in (5) as before and

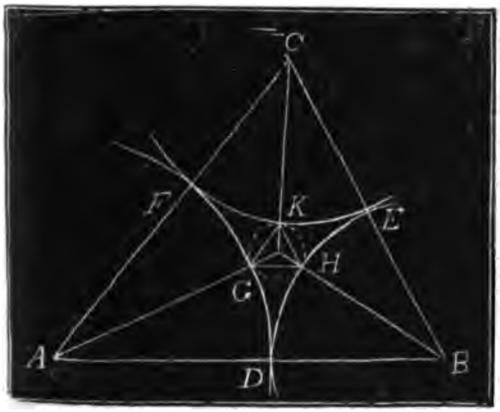
$$T_1 = 2\Big((n^2 - 1)Rr\Big)^{\frac{1}{2}} = \sqrt{(n^2 - 1)d_1 d_n}.$$

In this case, if $n = 2$, $\qquad T_1^2 = 3d_1 d_n$.

SOLUTION OF A PROBLEM.

BY E. B. SEITZ, GREENVILLE, OHIO.

Three circles whose radii are a, b, c, touch each other externally. Within the space enclosed by them a circle is drawn tangent to the three circles, and within this circle three circles are drawn tangent to each other and to the three given circles. Calling the radii of these three circles x_1, y_1, z_1, we may determine three other circles, radii x_2, y_2, z_2, touching each other and the second set of circles in a similar way; and so on. Find the radii, x_n, y_n, z_n, of the n^{th} set of inscribed circles.



SOLUTION.—Let A, B, C be the centers of the given circles, GHK the circle which touches them, O its center, A′, B′, C′ the centers of the first set of inscribed circles.

Put $AD = a$, $BD = b$, $CF = c$, $OG = r_1$, $A'G = x_1$, $B'H = y_1$, $C'K = z_1$, r_0 = the radius of the circle circumscribing the three given circles; also put $\angle BAC = \beta$, $\angle BAO = \theta$, $\angle CAO = \phi$, $\angle AOB = \omega$.

Then $\cos\beta = \frac{(a+b)^2 + (a+c)^2 - (b+c)^2}{2(a+b)(a+c)} = 1 - \frac{2bc}{(a+b)(a+c)}$,

$\cos\theta = 1 - \frac{2br_1}{(a+b)(a+r_1)}$, $\cos\psi = 1 - \frac{2cr_1}{(a+c)(c+r_1)}$, and

$\cos(\theta + \psi) = \cos\beta$, or by reduction

$$\cos^2\theta + \cos^2\psi - 2\cos\beta\cos\theta\cos\psi = 1 - \cos^2\beta \ldots\ldots\ldots\ldots\ldots (1).$$

Substituting the values of $\cos\beta$, $\cos\theta$, $\cos\psi$ in (1) and reducing, we have

$[4abc(a+b+c) - (ab+ac+bc)^2]r_1^2 + 2abc(ab+ac+bc)r_1$
$= a^2b^2c^2$; whence

$$r_1 = \frac{\pm 2abc\sqrt{abc(a+b+c)} - abc(ab+ac+bc)}{4abc(a+b+c) - (ab+ac+bc)^2}$$

$$= \frac{abc}{\pm 2\sqrt{abc(a+b+c)} + ab + ac + bc}.$$

The first of these values is the value of r_1, and the other is that of r_0, taken with a contrary sign;

$$\therefore r_1 = \frac{abc}{2\sqrt{abc(a+b+c)} + ab + ac + bc},$$

$$\text{and } r_0 = \frac{abc}{2\sqrt{abc(a+b+c)} - (ab+ac+bc)}.$$

$\cos\omega = 1 - \frac{2x_1y_1}{(r_1 - x_1)(r_1 - y_1)} = 1 - \frac{2ab}{(a+r_1)(b+r_1)}$; whence

$$\frac{x_1y_1}{(r_1 - x_1)(r_1 - y_1)} = \frac{ab}{(a+r_1)(b+r_1)} \ldots\ldots\ldots\ldots\ldots (2),$$

and similarly we find $\frac{x_1z_1}{(r_1 - x_1)(r_1 - z_1)} = \frac{ac}{(a+r_1)(c+r_1)} \ldots\ldots\ldots (3)$,

$$\frac{y_1z_1}{(r_1 - y_1)(r_1 - z_1)} = \frac{bc}{(b+r_1)(c+r_1)} \ldots\ldots\ldots\ldots\ldots (4).$$

From (2), (3), (4) we easily find

$\frac{x_1}{r_1 - x_1} = \frac{a}{a + r_1}$, $\frac{y_1}{r_1 - y_1} = \frac{b}{b + r_1}$, $\frac{z_1}{r_1 - z_1} = \frac{c}{c + r_1}$; whence

$x_1 = \dfrac{ar_1}{2a + r_1}$, $y_1 = \dfrac{br_1}{2b + r_1}$, $z_1 = \dfrac{cr_1}{2c + r_1}$. Similarly we find

$x_2 = \dfrac{x_1 r_2}{2x_1 + r_2}$, $y_2 = \dfrac{y_1 r_2}{2y_1 + r_2}$, $z_2 = \dfrac{z_1 r_2}{2z_1 + r_2}$; and so on.

Taking the reciprocals of the values of $x_1, x_2, x_3, x_4 \ldots . x_n$, we have

$$\frac{1}{x_1} = \frac{1}{a} + \frac{2}{r_1},$$

$$\frac{1}{x_2} = \frac{1}{x_1} + \frac{2}{r_2} = \frac{1}{a} + \frac{2}{r_1} + \frac{2}{r_2},$$

$$\frac{1}{x_3} = \frac{1}{x_2} + \frac{2}{r_3} = \frac{1}{a} + \frac{2}{r_1} + \frac{2}{r_2} + \frac{2}{r_3},$$

....

$$\frac{1}{x_n} = \frac{1}{x_{n-1}} + \frac{2}{r_n} = \frac{1}{a} + \frac{2}{r_1} + \frac{2}{r_2} + \frac{2}{r_3} + \frac{2}{r_4} + \ldots + \frac{2}{r_n} \ldots\ldots (5).$$

Taking the reciprocals of the values of r_1 and r_n, and subtracting, we obtain $\dfrac{1}{r_1} - \dfrac{1}{r_0} = \dfrac{2}{a} + \dfrac{2}{b} + \dfrac{2}{c}$ ....(6).

Similarly we find

$$\frac{1}{r_2} - \frac{1}{r_1} = \frac{2}{x_1} + \frac{2}{y_1} + \frac{2}{z_1} = \frac{2}{a} + \frac{2}{b} + \frac{2}{c} + \frac{12}{r_1} \ldots\ldots\ldots\ldots (7).$$

Subtracting (6) from (7), and transposing, we get

$\dfrac{1}{r_2} = \dfrac{14}{r_1} - \dfrac{1}{r_0}$; and similarly we find

$$\frac{1}{r_3} = \frac{14}{r_2} - \frac{1}{r_1},\quad \frac{1}{r_4} = \frac{14}{r_3} - \frac{1}{r_2}, \ldots\ldots \frac{1}{r_n} = \frac{14}{r_{n-1}} - \frac{1}{r_{n-2}}.$$

Substituting the values of $r_2, r_3, r_4 \ldots\ldots r_{n-1}$, taken from the preceding equations, and multiplying by $(7 + 4\sqrt{3}) + (7 - 4\sqrt{3})$ instead of 14, we have

$$\frac{1}{r_1} = \frac{\frac{1}{8\sqrt{3}}[(7+4\sqrt{3})-(7-4\sqrt{3})]}{r_1},$$

$$\frac{1}{r_2} = \frac{\frac{1}{8\sqrt{3}}[(7+4\sqrt{2})^2-(7-4\sqrt{3})^2]}{r_1} - \frac{\frac{1}{8\sqrt{3}}[(7+4\sqrt{3})-(7-4\sqrt{3})]}{r_0},$$

$$\frac{1}{r_3} = \frac{\frac{1}{8\sqrt{3}}\left[(7+4\sqrt{3})^3-(7+4\sqrt{3})^3\right]}{r_1} - \frac{\frac{1}{8\sqrt{3}}\left[(7+4\sqrt{3})^2-(7+4\sqrt{3})^2\right]}{r_0},$$

....

$$\frac{1}{r_n} = \frac{\frac{1}{8\sqrt{3}}\left[(7+4\sqrt{3})^n-(7-4\sqrt{3})^n\right]}{r_1}$$

$$-\frac{\frac{1}{8\sqrt{3}}\left[(7+4\sqrt{3})^{n-1}-(7-4\sqrt{3})^{n-1}\right]}{r_0}.$$

Adding these equations, we get

$$\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}+\dots\dots\dots\dots\frac{1}{r_n}$$

$$=\frac{\frac{1}{48}\left[(7+4\sqrt{3})^{n+\frac{1}{2}}+(7-4\sqrt{3})^{n+\frac{1}{2}}\right]}{r_1}$$

$$-\frac{\frac{1}{48}\left[(7+4\sqrt{4})^{n-\frac{1}{2}}+(7-4\sqrt{3})^{n-\frac{1}{2}}\right]}{r_0}$$

$$=\left\{\frac{1}{4\sqrt{3}}\left[(7+4\sqrt{3})^n-(7-4\sqrt{3})^n\right]\sqrt{abc(a+b+c)}+\frac{1}{12}\left[(7\right.\right.$$

$$\left.\left.+4\sqrt{3})^n+(7-4\sqrt{3})^n\right](ab+ac+bc)\right\}\div abc.$$

Substituting this value in (5), and taking reciprocals, we find

$$x_n = abc \div \left\{\frac{1}{2\sqrt{3}}\left[(7+4\sqrt{3})^n-(7-4\sqrt{3})^n\right]\sqrt{abc(a+b+c)}+\tfrac{1}{6}\left[(7+\right.\right.$$

$$\left.\left.4\sqrt{3})^n+(7-4\sqrt{3})^n-2\right](ab+ac)+\tfrac{1}{6}\left[(7+4\sqrt{3})^n+(7-4\sqrt{3})^n+4\right]bc\right\}.$$

Similarly we find

$$y_n = abc \div \left\{\frac{1}{2\sqrt{3}}\left[(7+4\sqrt{3})^n-(7-4\sqrt{3})^n\right]\sqrt{abc(a+b+c)}+\tfrac{1}{6}\left[(7+\right.\right.$$

$$\left.\left.4\sqrt{3})^n+(7-4\sqrt{3})^n-2\right](ab+bc)+\tfrac{1}{6}\left[(7+4\sqrt{3})^n+(7-4\sqrt{3})^n+4\right]ac\right\}.$$

$$z_n = abc \div \left\{ \frac{1}{2\sqrt{3}} [(7+4\sqrt{3})^n - (7-4\sqrt{3})^n] \sqrt{abc(a+b+c)} + \tfrac{1}{6}[(7+4\sqrt{3})^n + (7-4\sqrt{3})^n - 2](ac+bc) + \tfrac{1}{6}[(7+4\sqrt{3})^n + (7-4\sqrt{3})^n + 4)ab \right\}.$$

NOTE ON THE BINOMIAL THEOREM.

BY GAETANO LANZA, ASSITANT PROF. OF MATH. AND MECH., MASS. INSTITUTE OF TECNOLOGY.

MR. EDITOR:—Allow me to call your attention to a demonstration of the Binomial Formula for positive integral exponents, given by Bobillier, which, though very simple and lucid, is not very generally known.

It does not involve at all the subjects of Permutations and Combinations, and hence in order to understand the Binomial Theorem the student is not obliged previously to master a number of propositions on the above subjects.

It would therefore seem very desirable that this demonstration should find a place in our text books and be taught in our schools.

For the benefit of any of your readers who may not be familiar with the notation employed, I will state that $\underline{|n}$ (= *factorial n*) is an abbreviated way of writing $1.2.3.4 \ldots\ldots\ldots n$, that is the product of the natural numbers from 1 to n inclusive; thus $\underline{|3} = 1.2.3 = 6$; $\underline{|4} = 1.2.3.4 = 24$, &c.

The demonstration is as follows:

The following three equations will be evident on inspection, viz.:

$$\frac{(x+a)^1}{\underline{|1}} = \frac{x}{\underline{|1}} + \frac{a}{\underline{|1}},$$

$$\frac{(x+a)^2}{\underline{|2}} = \frac{x^2}{\underline{|2}} + \frac{x}{\underline{|1}} \cdot \frac{a}{\underline{|1}} + \frac{a^2}{\underline{|2}},$$

$$\frac{(x+a)^3}{\underline{|3}} = \frac{x^3}{\underline{|3}} + \frac{x^2}{\underline{|2}} \cdot \frac{a}{\underline{|1}} + \frac{x}{\underline{|1}} \cdot \frac{a^2}{\underline{|2}} + \frac{a^3}{\underline{|3}}.$$

It is now proposed first to find an expression equal to

$$\frac{(x+a)^n}{\underline{|n}}$$

where n is any positive integer. In the above three results the following laws held:

I. The number of terms on the right hand side is greater by one than the exponent of $(x + a)$ on the left hand.

II. The exponent of x in the first term is the same as the exponent of the power to which $(x + a)$ is raised, and in each succeeding term the exponent of x is less by unity than that in the preceding term; while that of a follows the same law beginning at the other end of the series.

III. Each power of x or of a has for denominator its exponent taken factorially.

We shall now prove that these laws hold whatever positive integral value be assigned to n.

Suppose the laws to hold when the exponent is $n - 1$, that is suppose

$$\frac{(x+a)^{n-1}}{\underline{|n-1}} = \frac{x^{n-1}}{\underline{|n-1}} + \frac{x^{n-2}}{\underline{|n-2}}\cdot\frac{a}{\underline{|1}} + \frac{x^{n-3}}{\underline{|n-3}}\cdot\frac{a^2}{\underline{|2}} + \dots\dots$$
$$\frac{x^{n-r-1}}{\underline{|n-r-1}}\cdot\frac{a^r}{\underline{|r}} + \dots + \frac{a^{n-1}}{\underline{|n-1}}.$$

Now multiply this identical equation by the following also identical, member by member, viz.:

$$\frac{x+a}{n} = \frac{x}{n} + \frac{a}{n},$$

and as the result of the multiplication we shall obtain the following:

$$\frac{(x+a)^n}{\underline{|n}} = \frac{x^n}{\underline{|n}} + \frac{x^{n-1}}{\underline{|n-1}}\cdot\frac{a}{\underline{|1}} + \frac{x^{n-2}}{\underline{|n-2}}\cdot\frac{a^2}{\underline{|2}} + \dots\dots\dots$$
$$+ \frac{x^{n-r}}{\underline{|n-r}}\cdot\frac{a^r}{\underline{|r}} + \dots\dots + \frac{a^n}{\underline{|n}} \quad (1).$$

Hence, if the above laws hold for an exponent $n - 1$, they hold for the exponent n, which is greater than the former by unity; but they have been proved to hold for the exponent 3, hence they hold for 4, hence for 5, and so on, ad infiinitum; therefore they are true in the case of any positive integral exponent.

Now multiply both members of equation (1) by $\underline{|n}$ and we obtain

$$(x+a)^n = x^n + nx^{n-1}a + \frac{n(n-1)}{1.2}x^{n-2}a^2 + \dots\dots\dots$$
$$+ \frac{\underline{|n}}{\underline{|n-r}\;\underline{|r}}x^{n-r}a^r + \dots\dots a^n,$$

which is the Binomial Formula.

SOLUTIONS OF PROBLEMS IN NO. 8.

Solutions of problems in No. 8 have been received as follows:

From Prof. A. B. Evans, 34, 35 & 36; R. M. DeFrance, Esq., 34; Henry Gunder, 34 & 35; L. Regan, 34 & 35; James Stott, 35; and Walter Siverly, 35 & 36.

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35. "Given $xyz = 18$, (1); $x^2 + y^2 + z^2 = 33$, (2); $(x^2 - yz)^3 + (y^2 - xz)^3 + (z^2 - xy)^3 - 3(x^2 - yz)(y^2 - xz)(z^2 - xy) = 6561$, (3); to find x, y and z."

SOLUTION BY R. M. DEFRANCE, ESQ., MERCER, PA.

Put $x + y + z = s$ and $xy + xz + yz = m$, and substitute in (3) and it becomes $(s^3 - 3ms + 54)^2 - 108(s^3 - 3ms + 54) = 3645$; whence we get $s^3 - 3ms = \pm 81$ ....(4).

By substitution (2) becomes $s^2 - 2m = 33$ ....(5).

Eliminating m from (4) and (5) we get $s^3 - 99s = \mp 162$. Multiplying this eq. by s we have $s^4 - 99s^2 = \mp 162s$, or, by transposition, $s^4 - 18s^2 = 81s^2 \mp 162s$. $\therefore$ $s^4 - 18s^2 + 81 = 81s^2 \mp 162s + 81$, whence $s^2 - 9 = 9s \mp 9$ and $s = 9$ or $\frac{1}{2}(9 \pm \sqrt{153})$ ....(6).

Substituting for s in (5) we get $m = 24$ or $\frac{1}{4}(51 \pm 9\sqrt{153})$.........(7).

Put $x + y = t$ and $xy = u$, and, neglecting the irrational values of s and m, we have $t + z = 9$, $u + tz = 24$ and $uz = 18$. Eliminating t and u from these three equations we get

$$z^3 - 9z^2 + 24z - 18 = 0 \text{.........(8).}$$

Put $z = v + 3$ and substitute in (8), we get $v^3 - 3v = 0$, $\therefore v^2 = 3$ and $v = \pm\sqrt{3}$. $\therefore z = 3 \pm \sqrt{3}$. But $x + y + z = 9$ (by (6)), $\therefore$ $x + y = 9 - (3 \pm \sqrt{3}) = 6 \mp \sqrt{3}$. From (1) $xy = 18 \div z = 18 \div (3 \pm \sqrt{3})$, $\therefore x^2 - 2xy + y^2 = (6 \mp \sqrt{3})^2 - 72 \div (3 \pm \sqrt{3}) = 3$. $\therefore$ $x - y = \pm\sqrt{3}$. But $x + y = 6 \mp \sqrt{3}$ $\therefore 2x = 6$ and $2y = 6 \mp 2\sqrt{3}$, and consequently $x = 3$, $y = 3 \mp \sqrt{3}$ and $z = 3 \pm \sqrt{3}$.

---

35. "A man let a stone weighing 40 lbs. to a neighbor—the latter broke it accidentally into four parts—and upon returning the fragments consoled the owner by remarking that now he could weigh all numbers

between one and forty. In other words, given $a + b + c + d = 40$, to determine such values for a, b, c and d as will, by *association*, produce all numbers from one to forty."

SOLUTION BY PROF. A. B. EVANS, LOCKPORT, N. Y.

It is evident that we must have $b - a = a + 1$, $c - b - a = a + b + 1$, $d - a - b - c = a + b + c + 1$. Therefore $b = 2a + 1$, $c = 2a + 2b + 1$, $d = 2a + 2b + 2c + 1$; and $a + b + c + d = 7a + 4b + 2c + 3 = 27a + 13 = 40$; $\therefore a = 1$, $b = 3$, $c = 9$, $d = 27$.

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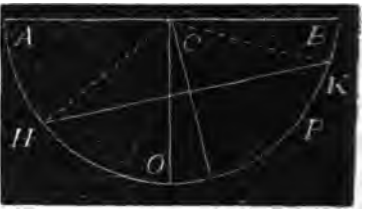
36. "A perfectly flexible cord of given length is suspended from two points whose coordinates are x', y' and x'', y''. How must the weight of the cord vary from point to point in order that it may hang in the arc of a circle."

SOLUTION BY WALTER SIVERLY, OIL CITY, PA.



Let H K be be the two given points of suspension, $\sqrt{(x' - x'')^2 + (y' - y'')^2} = 2b$ = the distance between them, $2l$ = the length of the cord, r the radius of the circle, $r\sin^{-1}\dfrac{b}{r} = l$,

from which r may be determined and the circle drawn passing through H and K. Let AB be the horizontal diameter of the circle, O its lowest point to which transfer the origin making the axis of x horizontal and of y vertical, and let m be the unit of mass at any point P; and c the tension at O. It is shown in works on statics that

$$m = \frac{c}{g}\frac{d^2y}{dx^2} \div \frac{ds}{dx} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

The equation to the circle is,

$$x^2 = 2ry - y^2, \text{ or } y = r - \sqrt{(r^2 - x^2)},$$

$$\frac{dy}{dx} = \frac{x}{\sqrt{r^2 - x^2}}, \quad \frac{d^2y}{dx^2} = \frac{r^2}{\sqrt{(r^2 - x^2)^3}},$$

$$\frac{ds^2}{dx^2} = 1 + \frac{dy^2}{dx^2} = \frac{r^2}{r^2 \quad x^2}, \quad \frac{ds}{dx} = \frac{r}{\sqrt{r^2 - x^2}}.$$

Hence from (1), $m = \dfrac{c}{g}\cdot\dfrac{r}{r^2 - x^2} = \dfrac{cr}{g\,(r - y)^2}$, or the unit of mass varies inversely as the square of its distance below the horizontal diameter of the circle.

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The Analyst:

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PURE AND APPLIED MATHEMATICS.

EDITED AND PUBLISHED BY

J. E. HENDRICKS, A. M.

CONTENTS.

DES MOINES, IOWA:
PIERSON & BLAIR, BOOK AND JOB PRINTERS.

NOTICE.—This No. completes Vol. 1 of THE ANALYST. We commenced the publication under the impression that a journal devoted specially to the science of mathematics would not only assist the student who is engaged in the study of mathematics, but would tend to promote the cultivation of that science which constitutes the foundation of all other sciences.

We were fully conscious, at the time, of the difficulties that would beset us in attempting to establish such a journal: The first and most important of which was our inexperience in conducting such an enterprise and our limited mathematical reading and information; secondly, the want of a suitable mathematical library, and thirdly, the want of facilities for properly illustrating and printing such a journal. We did not therefore anticipate pecuniary success in our undertaking; and, moreover, we were cautioned by our friends whose experience in such matters was greater than ours, that we should not expect a large subscription list. Our hope therefore in commencing the enterprise was mainly that we might continue the publication a year without very great pecuniary loss, and that, within that time, some party more favorably located and better able to conduct such an enterprise might be induced to *continue* the publication; of which however, at present, we see no prospect.

Our list of subscribers comprising college students and common school teachers is not as large as we had anticipated, whereas the number of college professors and eminent mathematicians throughout the country who have subscribed and paid liberally for our publication, is much larger than we had expected; which fact has induced us to believe that there is a very general desire among mathematicians to support and encourage even a feeble effort to establish a mathematical journal in this country.

Believing, therefore, that such defects as must necessarily result from the causes above enumerated will be generously ovelrooked by mathematicians, and trusting that we may receive a larger patronage from students and teachers during the next year than heretofore, we have concluded, though at some pecuniary sacrifice, to continue the publication another year at least, or until some suitable person can be induced to take it off our hands.

We have concluded to issue Vol. 2 in Nos. of 32 pages each, at intervals of *two months*, commencing about the 15th of Jan., 1875.

We hereby specially request all subscribers for Vol. 1 who do not intend to subscribe for Vol. 2 to notify us by letter or postal card before the 1st of January; and we hope our friends will send us as many *new* subscribers as they can before that time, also, so that we may judge how large an edition we may safely issue.

We have a limited number of complete sets of Vol. 1 which we will be pleased to send to purchasers, post paid, on receipt of $2, per set.

EDITOR.

THE ANALYST.

Vol. I. Nov. and Dec., 1874. Nos. 11 and 12.

THE USE OF IMAGINARY QUANTITIES IN ANALYTICAL GEOMETRY.

BY WERNER STILLE, MARINE, ILL.

In works on Analytical Geometry we often find the statement that for certain values of the argument x the function y becomes imaginary and hence cannot be constructed. Yet at the same time we find that the function, although imaginary, still varies, still remains a function of x. For example take the central equation of the ellipse:

$$y=\frac{b}{a}\sqrt{a^2-x^2}\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (1).$$

Here y becomes imaginary for $x>a$, and yet y continues to vary as x continues to increase. The question therefore arises whether our notation is not deficient. If our notation were well adapted to represent the relations between the argument and its function, then we should be able to construct that function for *all* values of x.

By Gauss' interpretation of the imaginary numbers we know that when the "real" numbers are conceived as situate upon a straight line, then the "imaginary" numbers *must* be taken as situate upon a line right-angular to the former and having the point zero in common with it. Hence for the purposes of the Analytical Geometry of the Plane, it may be well to introduce imaginary numbers.

Let us see what notation would result from the introduction of these numbers and what advantages may be derived from it. An equation between x and y, such as equation (1), states what shall be the relative *lengths* of the abscissa and the ordinate of any point of the curve; but nothing is expressed as to the relative *directions* in which x and y shall

be counted. Therefore if, according to usage, we tacitly add the condition that y shall be perpendicular upon x, we need not be surprised if occasionally we find a construction impossible. If we wish to have the ordinates perpendicular to the abscissas, we should express this in some way. This can be conveniently done by the well known formula

$$z = x + i.y \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

z denoting the position of any point of the curve, x its abscissa, y its ordinate and $i = \sqrt{-1}$, which symbol expresses that y is perpendicular upon x. Now eliminating y from (1) and (2) we have

$$z = x + i\,\frac{b}{a}\sqrt{a^2 - x^2} \ldots\ldots\ldots\ldots\ldots (3).$$

Here z is tho position of any point of the curve, x the abscissa and $\frac{b}{a}\sqrt{a^2 - x^2}$ the ordinate standing perpendicular upon the x axis. The curve so generated is an ellipse, as is well known.

Now let us see what becomes of the movable point z when x becomes greater than a. Put $x = a + t$, then

$$z = a + t + i\,\frac{b}{a}\sqrt{-(2at + t^2)} = a + t + i^2\,\frac{b}{a}\sqrt{2at + t^2}.$$

But $i^2 = -1$, hence $z = a + t - \frac{b}{a}\sqrt{2at + t^2} \ldots\ldots\ldots\ldots (4).$

Now z, which was before of the form $\zeta + i\eta$, has become a "real" number, that is to say, z has entered the x — axis and is still a function of x as shown by equation (4), siuce $x = a + t$. The point z moves on the x axis. More strictly there are two points z, as in fact there were before x became greater than a. The ambiguity of $\sqrt{2at + t^2}$ shows that two such points z exist; and their rate of motion as depending upon t is given by equation (4).

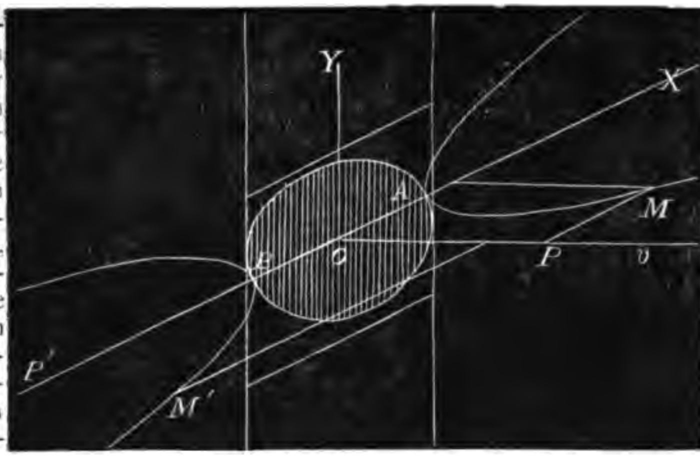
But since all numbers, real and imaginary, occupy only two dimensions, there is in space one dimension at our disposal and nothing prevents us from constructing equation (4) in a plane at right angles to the $x\,y$ plane.

This equation (4), as we readily see, represents an hyperbola when $a + t$, as before, is the abscissa and $\frac{b}{a}\sqrt{2at + t^2}$ the ordinate. Now

we can construct equation (1) or its equivalent by our present notation, namely equation (3)

$$z = x \pm i\,\frac{b}{a}\sqrt{a^2 - x^2}.$$

This equation gives an ellipse for $x<a$ and an hyperbola for $x>a$. The plane in which the hyperbola lies, is perpendicular upon the plane in which the ellipse is situated. Let xoy be a perpendicular plane, xov a horizontal plane, then the ellipse lies in the vertical plane and the corresponding hyperbola in the horizontal plane.



The equation of the hyperbola

$$y = \frac{b}{a}\sqrt{x^2 - a^2}$$

gives an ellipse for the values of x lying between $+a$ and $-a$: and an hyperbola for values of x absolutely greater than a. For, again,

$$z = x \pm i\,\frac{b}{a}\sqrt{x^2 - a^2}; \qquad \text{put } x = a - t$$

and we have, similarly to equation (4),

$$z = a - t \pm i\,\frac{b}{a}\sqrt{-2at + t^2} = a - t \pm i^2\,\frac{b}{a}\sqrt{2at - t^2}$$

$$z = a - t \mp \frac{b}{a}\sqrt{2at - t^2},$$

which is the well known vertex-equation of the ellipse, as is evident when recollecting that $a-t$ is the abscissa and $\frac{b}{a}\sqrt{2at - t^2}$ the ordinate.

The *same* figure therefore serves to show the geometrical meaning of the two equations $y = \frac{b}{a}\sqrt{a^2 - x^2}$, and $y = \frac{b}{a}\sqrt{x^2 - a^2}$.

In the same manner we find that the equation of the circle

$$y = \sqrt{a^2 - x^2}$$

represents an hyperbola for $x > a$; and that the equation of the parabola $y = \sqrt{ax}$ represents another parabola situated in a plane perpendicular upon the xy plane.

From what has been said it is evident that our notation will enable us at all times to construct the function when the ordinates are imaginary. In fact, the geometrical sense of an imaginary number being that it is to be counted in a direction perpendicular upon the line of real numbers - we see that the symbol i may be taken as nothing but a *coefficient of direction.*

By the ordinary method the equation $\quad x^2 + y^2 + a^2 = 0$

cannot be constructed; for $\quad y = \sqrt{-(a^2 + x^2)}$,

which shows that y is imaginary for all positive and negative values of x. But it is evident that y is a continuous function of x, and that therefore we ought to be able to construct corresponding values of x and y into some curve. Again employing our notation,

$$z = x \pm i\sqrt{-(a^2 + x^2)} = x \mp \sqrt{a^2 + x^2},$$

which gives an hyperbola.

After these introductory remarks I proceed to some more general considerations. Any curve in the xy plane may be represented by the symbol

$$z = \zeta + i\eta \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5),$$

ζ being a function of x; and η another function of x. This method is preferable to putting $\quad z = x + iy$,

where x and y are the coordinates, since by the former method functions can often be written in a more tractable form, than by the latter.

Since ζ is the abscissa and η the length of the ordinate of any point of the curve, therefore the tangent at any point of the curve will form

an angle θ with the ζ-axis whose trigonometrical tangent will be $\frac{d\eta}{d\zeta}$, *i. e.*,

$$\tan\theta = \frac{d\eta}{d\zeta}.$$

We also see that the well known expressions for the lengths of the normal N, the subnormal S_n, the tangent T, and the subtangent S_t will remain unchanged in form, hence

$$N = \eta\sqrt{1 + \left(\frac{d\eta}{d\zeta}\right)^2}; \qquad S_n = \eta\,\frac{d\eta}{d\zeta}.$$

$$T = \eta\sqrt{1 + \left(\frac{d\zeta}{d\eta}\right)^2}; \qquad S_t = \eta.\,\frac{d\zeta}{d\eta}.$$

In short, all the relations between the differential coefficients and the form of the curve are immediately applicable to our present notation.

We will now discuss the geometrical meaning of some analytical relations known to exist between real and imaginary functions. A function of the form $f(x + iy)$ we will call (as has become customary) a *complex function;* a number of the form $x + iy$ a *complexe number.*

I. From the theory of complex functions we know that

$$\sin(x + iy) = \sin x.\cos(iy) + \cos x.\sin(iy).$$

Or, introducing for $\cos(iy)$ and $\sin(iy)$ their exponential equivalents,

$$\sin(x + iy) = \sin x.\left(\frac{e^y + e^{-y}}{2}\right) + i.\cos x\left(\frac{e^y - e^{-y}}{2}\right) \dots\dots\dots\dots (6).$$

Now x and y are independent of each other unless we establish some functional relation between these two variables. Hence this equation expresses a very general relation and will admit of the construction of a very great number of curves. Let us then assign some special value to y. At first let y be a constant number $y = a$, then (6) becomes

$$\sin(x + ia) = \sin x.\left(\frac{e^a + e^{-a}}{2}\right) + i.\cos x.\left(\frac{e^a - e^{-a}}{2}\right) \dots\dots\dots\dots (7),$$

which for the sake of brevity, we may write thus

$$\sin(x + ia) = A.\sin x + i.B.\cos x,$$

so that, comparing this with the general formula $z = \zeta + i\eta$, we see that

$$\zeta = A.\sin x; \qquad \eta = B.\cos x.$$

Now in order to find what curve is represented by (7) let ur express η as a function of ζ. Since

$$\zeta^2 = A^2.\sin^2 x; \qquad \eta^2 = B^2.\cos^2 x, \text{ we find}$$

$$\eta^2 = B^2\left(1 - \frac{\zeta^2}{A^2}\right), \text{ hence}$$

$$A^2\eta^2 + B^2\zeta^2 = A^2B^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8).$$

This we recognize at once as the central equation of the ellipse. No. (7) therefore is the equation of an ellipse whose major axis A $= \dfrac{e^a + e^{-a}}{2}$ and whose minor axis B $= \dfrac{e^a - e^{-a}}{2}$. Let $a = \infty$ then B $=$ A, and the curve becomes a *circle*. For $a = 0$ the imaginary part vanishes and therefore the ellipse degenerates into a *straight line*.

These results may be expressed in the following

Theorem: The equation $\qquad z = \sin(x + ia)$

represents an ellipse whose major axis is $\dfrac{e^a + e^{-a}}{2}$ and whose minor axis is $\dfrac{e^a - e^{-a}}{2}$.

The *construction* will be this: Since the argument is $x + ia$, the imaginary part of it is constant and x describes a straight line parallel to the x-axis and at a distance from it equal to unity, namely the line in the figure marked $1\ i$, $i + x$. When x is 1, 2, 3, &c., then the argument is $1 + i$, $2 + i$, $3 + i$, &c. At the same time that the argument describes this straight line, the movable point z (the function) describes the ellipse. The values of the function belonging to any given value of the argument are calculated after reducing the function to the form $\zeta + i\eta$. In the present case

$$z = \tfrac{1}{2}(e^a + e^{-a}).\sin x + i\tfrac{1}{2}(e^a - e^{-a}).\cos x.$$

II. In No. (6) let x be constant, $x = a$, while y varies, then

$$\sin(a + iy) = \sin a\left(\frac{e^{y} + e^{-y}}{2}\right) + i.\cos a\left(\frac{e^{y} - e^{-y}}{2}\right).$$

Now in $z = \zeta + i\eta$ we have $\zeta = \sin a\left(\frac{e^{y} + e^{-y}}{2}\right)$ and

$\eta = \cos a\left(\frac{e^{y} - e^{-y}}{2}\right)$; hence to express η as a function of ζ,

$$\zeta^2 = \frac{\sin^2 a}{4}\,(e^{2y} + 2 + e^{-2y}),$$

$$\eta^2 = \frac{\cos^2 a}{4}\,(e^{2y} - 2 + e^{-2y}),$$

$$\eta^2 = \frac{\cos^2 a}{2}\left(\frac{4\zeta^2}{\sin^2 a} - 4\right),$$

$$\sin^2 a.\eta^2 = \cos^2 a.\zeta^2 - \sin^2 a\cos^2 a \ldots\ldots\ldots\ldots\ldots\ldots\ldots (9).$$

This is the well known central equation of the *hyperbola*. For $a = 0$ we obtain a straight line parallel to the axis of ordinates whose distance from it is $\frac{1}{2}(e - \frac{1}{e})$.

For $a = \pi \div 2$ the curve becomes a similar straight line parallel to the x-axis.

Construction: The argument moves on a line which is parallel to the η-axis and at a distance from it equal to unity. This straight line passes through the intersection of the axes of the hyperbola, which point is in the x-axis at the distance $= 1$ from the origin of the coordinates. The abscissas of the hyperbola are real, the ordinates imaginary.

We now have this *theorem:*

The formula $z = \sin(a + iy)$ is the equation of an hyperbola whose axes are $\sin a$ and $\cos a$.

III. In No. (6) let $y = x$, then

$$\sin(x + ix) = \sin x\left(\frac{e^{x} + e^{-x}}{2}\right) + i.\cos x\left(\frac{e^{x} - e^{-x}}{2}\right) \ldots\ldots\ldots (10).$$

Now the argument $x + ix$ describes a straight line forming an angle of $45°$ with the x-axis, while the function describes a curve in the plane of these two lines. For $x = \pi, 2\pi, 3\pi$, &c., the abscissa vanishes, because then $\sin x = 0$, while the factor $\frac{e^{x} + e^{-x}}{2}$ increases very rapidly. For $x = \frac{1}{2}\pi, \frac{3}{2}\pi, \frac{5}{2}\pi$, &c., the ordinate vanishes while the fator $\frac{e^{x} - e^{-x}}{2}$ in-

creases very rapidly. The curve therefore is a kind of spiral. Its origine is in the origine of coordinates.

Differentiating No. (10) we have

$$\frac{d\eta}{d\zeta} = \frac{\cos x(e^x + e^{-x}) - \sin x(e^x - e^{-x})}{\sin x(e^x - e^{-x}) + \cos x(e^x + e^{-x})}.$$

When $\sin x = 0$, then the abscissa $= 0$; but for $\sin x = 0$, $d\eta \div d\zeta = 1$, therefore the curve at every intersection with the η-axis forms an angle of 45° with that axis. Again, for $\cos x = 0$ the ordinate $= 0$; but for $\cos x = 0$, $d\eta \div d\zeta = -1$; therefore the curve at every intersection with the ζ-axis forms with that axis an angle of 45°.

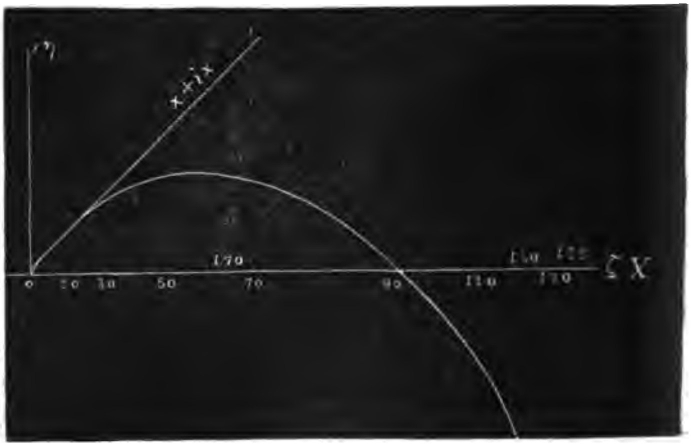
In order that $d\eta \div d\zeta$ may be $= 0$, we must have

$$\cos x(e^x + e^{-x}) - \sin x(e^x - e^{-x}) = 0;$$

hence
$$\tan x = \frac{e^x + e^{-x}}{e^x - e^{-x}}.$$

The expression on the right hand side of this equation shows that as x increases, the value of $\tan x$ approaches rapidly to unity. Hence $d\eta \div d\zeta$ will become $= 0$ for $\tan x = 1$, that is to say for $x = \frac{1}{4}\pi, \frac{3}{4}\pi, \frac{5}{4}\pi$, &c. For $x = \frac{1}{4}\pi$ the approximation begins, but for $\frac{3}{4}\pi, \frac{5}{4}\pi$, &c., t increases very rapidly.

The construction is given ir the figure. Bu the curve soor assumes suct enormous proportions that i has been tracec only a little beyond $x = 135°$ The argumen describes th straight line o



$x + ix$. On the axis of abscissas there are marked the points corresponding to $x = 20, 30, 50$, &c., degress.

We have thus far considered only one complex function, namely $\sin(x + iy)$, and we see that it will produce a very great number of

curves, according to the relation which we choose to establish between x and y. The same is true of all other complex functions.

To recapitulate: The introduction of the imaginary numbers into Analytical Geometry enables us to construct *all* functions of the abscissa, whether real or imaginary. For example the equations

$$y^2 + x^2 + 4y - 2x + 8 = 0,$$
$$y^4 + 3y^2 + x^2 - 5x + 10 = 0, \text{ \&c., \&c.,}$$

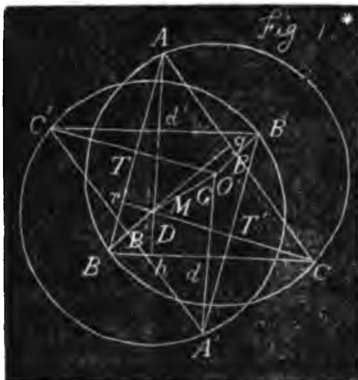
without the introduction of imaginary numbers cannot be constructed; but when *expressing* the condition that y is to be perpendicular upon x, it is easy to construct them. And indeed they must each represent some curve since y varies when x does.

THE PLANE TRIANGLE AND ITS SIX CIRCLES.

BY ASHER B. EVANS, A. M., LOCKPORT, N. Y.

The six circles whose properties are discussed in this article are *the circumscribed*, *the inscribed*, *the nine-point*, and *the three escribed circles*. The first two of these circles are familiar to every student of elementary geometry. The *nine-point circle* in a triangle is that circle whose circumference passes through the feet of the three perpendiculars from the angles upon the opposite sides, the three middle points of the sides, and the three middle points of the segments of the perpendiculars between the angles and their common point of meeting. *The escribed circles* are three circles situated wholly without the triangle, each of which is tangent to one side of the triangle and to the other two sides produced.

Three points being in general sufficient to determine a circumference, it is necessary to show that the nine points enumerated in the definition of the nine-point circle are always on the same circumference. To this end let ABC (Fig. 1) be a triangle, O' the centre of its circumscribed circle, $O'\alpha$, $O'\beta$, $O'\gamma$ the perpendiculars from O' to the sides BC, AC, AB, respectively. Produce $O'\alpha$ to A', $O'\beta$ to B', $O'\gamma$ to C', making $\alpha A' = O'\alpha$, $\beta B' = O'\beta$, $\gamma C' = O'\gamma$; complete the triangle $A'B'C'$, let L be the centre of its circumscribed circle, and let α', β', γ' be the intersections of AL, BL, CL with $B'C'$, $A'C'$, $A'B'$.



1. The triangles ABC, A'B'C' are equal and their corresponding sides are parallel.

For γ being the middle point of AB and O'C', and β the middle point of AC and O'B', the three lines BC, $\gamma\beta$, C'B' are parallel and BC $= 2\gamma\beta =$ C'B'. Similarly AC $= 2\gamma\alpha =$ C'A' and AB $= 2\beta\alpha =$ B'A'.

2. The triangles $\alpha\beta\gamma$, $\alpha'\beta'\gamma'$ are equal and their corresponding sides are parallel.

Since BO' = BC' = C'A = C'L = L'B. α' is the middle point of C'B'. Similarly β' and γ' are the middle points of C'A' and A'B'.

3. L and O' are the intersections of perpendiculars of the triangles ABC and A'B'C'.

For AL, BL, CL being perpendicular to C'B', C'A', A'B', are perpendicular to BC, AC, AB (1); and A'O', B'O', C'O' being perpendicular to BC, AC, AB are perpendicular to C'B', C'A', A'B'.

4. The triangles ABC, A'B'C' are reciprocal.

For O', L and α, α' being homologous points in these two triangles, A'O' = AL, αO' $=$ αL; and therefore A'α = Aα' and Aα' = α'L' Similarly Bβ' = β'L and Cγ' = γ'L.

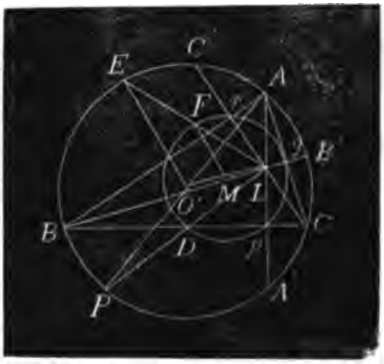
5. If a circumference be described with M the middle point of LO' as centre and ½O'C as radius, it will circumscribe the triangles $\alpha\beta\gamma$, $\alpha'\beta'\gamma'$ and pass through p, q, r, the feet of the perpendiculars of the triangle ABC.

For the similar triangles O'Mα, O'LA' give $\dfrac{\text{A'L}}{\alpha\text{M}} = \dfrac{\text{LO'}}{\text{MO'}}$, $\therefore \alpha$M $= \frac{1}{2}$A'L $= \frac{1}{2}$O'C. Similarly βM $= \frac{1}{2}$O'C and γM $= \frac{1}{2}$O'C. Since $\alpha\alpha'$, $\beta\beta'$, $\gamma\gamma'$ are diagonals of the parallelograms αLα'O', βLβ'O', γLγ'O', they are bisected in M by the common diagonal O'L $\therefore \alpha$M $= \alpha'$M $= \beta$M $= \beta'$M $= \gamma$M $= \gamma'$M $= \frac{1}{2}$O'C. Moreover L being the intersection of perpendiculars (3), and $\alpha\alpha'$, $\beta\beta'$, $\gamma\gamma'$ diameters of the circumference circumscribing $\alpha\beta\gamma$ and $\alpha'\beta'\gamma'$, this circumference passes through p,

*The engraver has made several mistakes in lettering Fig. 1. For d and d' read α and α', for B and B' within the triangle read β and β', for h read p, for T and T' read γ and γ', and for D read L.—Ed.

q and r, the feet of the perpendiculars, and is therefore the circumference of the nine-point circle of the triangle ABC.

6. The circumference of the nine-point circle bisects the straight line joining any point in the circumference of the circumscribing circle and the intersection of the perpendiculars.



For E (Fig. 2) being any point on the circumference of the circumscribing circle, and MF being drawn parallel to O′E and cutting EL in F,
$\frac{O'E}{MF}=\frac{LE}{LF}=\frac{LO'}{LM}=2.$ ∴ the radius of the nine-point circle being ½O′E, F the middle point of LE, is on the circumference of the nine-point circle.

7. If the perpendiculars Ap, Bq, Cr, produced, meet the circumference of the circumscribing circle in A′, B′, C′, the lines LA′, LB′, LC′ will be bisected by BC, AC, AB, respectively.

For the circumference of the nine-point circle passes through p, q, r, and bisects LA′, LB′, LC′ (6).

8. If the diameter AO′P be drawn, PL and BC will bisect each other.

For D being the middle point of PL, O′D is parallel to AL and hence perpendicular to BC. Moreover the foot of the perpendicular from O′ upon BC and the middie point of PL are both upon the circumference of the nine-point circle; therefore D is on BC and the truth of the theorem is evident.

9. Four circles may be described, each of which shall touch the three sides of a plane triangle or those sides produced. If six straight lines be drawn joining the centres of these circles, two and two, prove that the middle points of these six lines are in the circumference of a circle circumscribing the given triangle.

(The figure can readily be supplied by the student.)

Let ABC be the triangle; O, O_1, O_2, O_3, the centres of the inscribed and three escribed circles. Then since O_1A, O_2B, O_3C are the perpendiculars of the triangle $O_1O_2O_3$ and O their intersection, the circle circumscribing ABC is the nine-point circle of the triangle $O_1O_2O_3$;

therefore the circumference of this circle bisects the six lines OO_1, OO_2, OO_3, O_1O_2, O_2O_3, O_1O_3.

10. R and r being the radii and O′ and O the centres of the circumscribed and inscribed circles, and L the intersection of perpendiculars,

$$(O'L)^2 - 2(OL)^2 = R^2 - 4r^2.$$

Since (Fig. 3) $\angle O'AL = O'AB - LAB = (B - C)$ and $\angle OAL = OAB - LAB = \frac{1}{2}(B - C)$ the triangles O′AL, OAL give

$(O'L)^2 = (AO')^2 + (AL)^2 - 2AO'.AL\cos(B - C)$ and

$(OL)^2 = (AO)^2 + (AL)^2 - 2AO.AL\cos\frac{1}{2}(B - C)$; or, since $r = AO\sin\frac{1}{2}A$, $AL = 2R\cos A$ ($AL = A'O' = 2O'a = 2R\cos A$ as is seen in Fig. 1) and $r = 4R\sin\frac{1}{2}A\sin\frac{1}{2}B\sin\frac{1}{2}C$ (Chauvenet's Trig., Eq. 298),

$$(O'L)^2 = R^2 - 4R^2[\cos(B - C) - \cos A]\cos A = R^2 - 8R^2\cos A\cos B\cos C \text{ and}$$

$$(O'L)^2 = \frac{r^2}{\sin^2\frac{1}{2}A} + 4R^2\cos^2 A - \frac{4Rr\cos A\cos\frac{1}{2}(B - C)}{\sin\frac{1}{2}A}$$

$$= \frac{2r^2}{1 - \cos A} + 4R^2\cos^2 A - \frac{4Rr\cos A}{1 - \cos A}[2\sin\frac{1}{2}A\cos\frac{1}{2}(B - C)]$$

$$= 2r^2 + 4R^2\cos^2 A + \frac{2r^2\cos A}{1 - \cos A}\left\{1 - \frac{4R}{r}\sin\frac{1}{2}A\cos\frac{1}{2}(B - C)\right\}$$

$$= 2r^2 - 4R^2\cos A(\sin B\sin C - \cos A)$$

$$= 2r^2 - 4R^2\cos A\cos B\cos C$$

$$\therefore (O'L)^2 - 2(OL)^2 = R^2 - 4r^2.$$

11. R and r_1 being the radii and O and O_1 the centres of the circumscribed and one of the escribed circles, and L the intersection of perpendiculars $\qquad (O'L)^2 - 2(O_1L)^2 = R^2 - 4r_1^2$.

Since (Fig. 3) $\angle O_1AL = \angle OAL = \frac{1}{2}(B - C)$, we have from the triangle O_1AL, $(O_1L)^2 = (AO_1)^2 + (AL)^2 - 2AO_1.AL\cos\frac{1}{2}(B - C)$; or, since $r_1 = AO_1\sin\frac{1}{2}A$, $AL = 2R\cos A$, and $r_1 = 4R\sin\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C$ (Chauvenet's Trig., Eq. 298),

$$(O_1L)^2 = \frac{r_1^2}{\sin^2\frac{1}{2}A} + 4R^2\cos^2 A - \frac{4Rr_1\cos A\cos\frac{1}{2}(B - C)}{\sin\frac{1}{2}A}$$

$$= \frac{2r_1^2}{1 - \cos A} + 4R^2\cos^2 A - \frac{4Rr_1\cos A}{1 - \cos A}[2\sin\frac{1}{2}A\cos\frac{1}{2}(B - C)]$$

$$= 2r_1^2 - 4R^2\cos A(\sin B\sin C - \cos A)$$

$$= 2r_1^2 - 4R^2\cos A\cos B\cos C.$$ Since by (11)

$(O'L)^2 = R^2 - 8R^2\cos A\cos B\cos C$, we have

$$(O'L)^2 - 2(O_1L)^2 = R^2 - 4r_1^2.$$

12. The nine-point circle is tangent to the inscribed circle internally.

The centre of the nine-point circle being at M (Fig. 3) the middle point of LO′, the triangle LOO′ gives

$$2(OL)^2 + 2(OO')^2 = 4(OM)^2 + (O'L)^2.$$

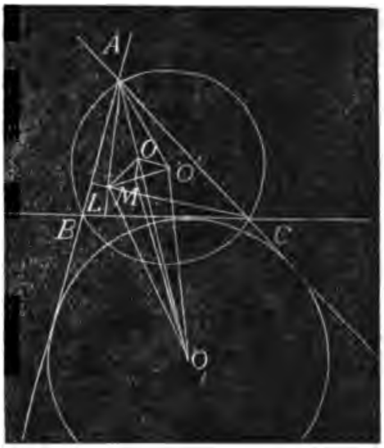
Since (10)

$(O'L)^2 - 2(OL)^2 = R^2 - 4r^2$ and

$(OO')^2 = R^2 - 2Rr$ (Chauvenet's Trig., Eq. 300),

$4(OM)^2 = R^2 - 4Rr + 4r^2$, and

$\therefore OM = \frac{1}{2}R - r.$



The radius of the nine-point circle being $\frac{1}{2}R$, the truth of the proposition is evident.

13. The nine-point circle is tangent to each of the three escribed circles externally.

In this case the triangle LO_1O' (Fig. 3) gives

$$2(O_1L)^2 + 2(O_1O')^2 = 4(O_1M)^2 + (O'L)^2.$$

Since (11) $(O'L)^2 - 2(O_1L)^2 = R^2 - 4r_1^2$ and $(O_1O')^2 = R^2 + 2Rr^1$ (Chauvenet's Trig., Eq. 301),

$$4(O_1M)^2 = R^2 + 4Rr_1 + 4r_1^2, \text{ and}$$

$\therefore O_1M = \frac{1}{2}R + r_1.$ Similarly, r_2, r_3 being the radii and O_2, O_3 the centres of the other two escribed circles,

$$O_2M = \frac{1}{2}R + r_2$$

and $$O_3M = \frac{1}{2}R + r_3.$$

The distances between the centre of the nine-point circle and the centres of the escribed circles show that the nine-point circle is tangent to each of the escribed cireles and moreover that the contact is external.

14. The nine-point circle of the triangle ABC (Fig. 1) is tangent to

the thirty-two inscribed and escribed circles of the triangles ABC, ALC, BLC, ALB, A′B′C′, A′O′C′, B′O′C′, A′O′B′.

Since α, β, γ, α', β', γ', the middle points of the sides of these eight triangles, are on the circumference of the nine-point circle of the triangle ABC, this nine-point circle is common to the eight triangles, and it is therefore tangent to their eight inscribed and twenty-four escribed circles.

15. The intersection of perpendiculars, the centre of the nine-point circle, the centre of gravity of the triangle, and the centre of the circumscribed circle, are four points in the same straight line, and their distances apart are in a constant ratio.

Let these four points be designated by L, M, G, O′ (Fig. 1). As Aα, Bβ, Cγ intersect in the centre of gravity of the triangle ABC, G is the centre of similitude of ABC and $\alpha\beta\gamma$. It is evident that M, L are the centres of similitude of $\alpha'\beta'\gamma'$ and $\alpha\beta\gamma$, ABC and $\alpha'\beta'\gamma$, respectively. Since G, M, L are the three centres of similitude of the three triangles ABC, $\alpha\beta\gamma$, $\alpha'\beta'\gamma'$ whose sides are respectively parallel, these three points are on the same straight line (Hackley's Geometry, Appendix 2, p. 2): but LM passes through O′; therefore the four points G, M, L, O′ are colinear.

L and O′ being the intersections of perpendiculars, and G the centre of similitude of the triangles ABC, $\alpha\beta\gamma$, we have

$$LG : GO' :: AB : \alpha\beta :: 2 : 1.$$

Therefore, since M is the middle point of LO′,

$$\frac{LM}{LO'} = \frac{1}{2}, \quad \frac{MG}{LO'} = \frac{1}{6}, \quad \frac{GO'}{LO'} = \frac{1}{3}.$$

The nine-point circle is celebrated in the history of geometry. Some of its properties were given by a German geometrician as early as 1822. Steiner gave the property of contact with the inscribed and escribed circles in 1828 in *Gergonne's Annales*. The designation "*nine-point*" was first given to this circle by Terquam in 1842, in the *Nouvelles Annales des Math.* This interesting circle began to attract the attention of English mathematicians about twenty years ago, and its properties have been given in various prize questions proposed at the English Universities; but I am not aware that any American mathematician has treated of its most simple properties.

A NEW METHOD OF FINDING DIFFERENTIALS—A SUGGESTION FOR MATHEMATICIANS.

BY PROFESSOR W. D. WILSON, OF THE CORNELL UNIVERSITY.

It seems to me that much of the obscurity and uncertainty now attending the first stages in the study of the Calculus can be avoided by means of the following suggestions:

The CALCULUS is a method of finding values by means of *rates of variation.*

A DIFFERENTIAL COEFFICIENT is any number by which it is necessary to multiply the rate at which one variable changes in order that it may be equal to the rate of variation of another.

Now suppose we have a variable as x, which increases *uniformly* at the rate of one unit of the measure of increase in one unit of the measure of time; we write the differential dx. On this supposition x becomes consecutively 1, 2, 3, 4, 5, &c.—and $dx = 1$. Now suppose we have $dy = 2$. We have another series 1, 3, 5, 7, &c., or possibly 0, 2, 4, 6, &c.

Suppose we have x and y and $u = x + y$; what is the differential of u, the function of $x + y$, or the sum of two variables?

Suppose $dx = 1$, as in the former case, and $dy = 2$, then we have, denoting the successive moments by the letters a, b, c, etc.,

	a,	b,	c,	d.			
$x = 1$,	2,	3,	4,	5,	etc.,	successively, and	$dx = 1$.
$y = 1$,	3,	5,	7,	9,	etc.,	"	$dy = 2$.
Sums, 2,	5,	8,	11,	14,		"	

Here, we see at once that the differential of the sum, $x + y$, is 3, or the sum of the differentials 1 and 2, or $du = dx + dy$. Or if we subtract one series from the other, term for term, we find in the same way that the differential of the difference of two variables, is the difference of their differentials.

Or if two horses of unequal strength, one of which can raise a weight at the rate of one mile an hour, and the other at the rate of two miles an hour, both pulling together they will raise it at the rate of three miles an hour. Or if they pull in opposite directions, they will raise it one

mile an hour in the direction of the effort of the strongest horse; we have, therefore, $d(x + y) = dx + dy$ and $d(x - y) = dx - dy$.

Now let us try the product of two variables, x and y; we have $u = xy$. As before, let a, b, c, etc., denote different moments of time, and we have moments,

		a,	b,	c,	d.	
$x =$	1,	2,	3,	4,	5,	etc., successively, and $dx = 1$.
$y =$	1,	3,	5,	7,	9,	" " $dy = 2$.
Products $=$	1,	6,	15,	28,	45,	"
Differences $=$	5,	9,	13,	17.		

If we take the values of one variable, as x at the beginning and at the end of any moment, as d, for example, add them together and divide the sum by two, we get the average of increase for that moment. If now we treat the values of the other variable for that moment in the same way, and then multiply the average of each variable by the differential of the others and add the sums, we get the same result as before, thus:

$$4 + 5 = 9 \text{ divided by } 2 = 4\tfrac{1}{2}.$$
$$7 + 9 = 16 \quad \text{"} \quad \text{by } 2 = 8.$$
$$\text{Now, } 4\tfrac{1}{2} \times 2 = 9 \text{ and } 8 \times 1 = 8.$$

And $8 + 9 = 17$, which is the increment of the product of the variables for that moment, or the rate is 17 divided by 1.

Nor is it at all necessary that the variables should increase at any uniform rate, the result will be the same whatever may be the rate or law of increase of the several factors.

If, in the cases before us, $x = y$ and $dx = dy$, then $xdy + ydx$ becomes $2xdx$, which is the differential expression for x^2. Or if we have instead, three variables, x, v and z, equal to each other, and their differentials are also respectively equal, we have the differential of a cube, or $d(x^3) = 3x^2dx$.

After having found the differential of the product of two variables, it is a much simpler and easier way to find that of three or more algebraically. Thus, for xyz let $s = xy$, then $sz = xyz$, and differentiating sz by the formula for the produst of two variables, we have $sdz + zds$. Substituting for s its value, xy, and for ds its value, $xdy + ydx$, and we shall have the result, $xydz + zydx + zxdy$.

And in fact with the two already given, namely, the sum and differ-

ence and the product of two variables, we can find all the others of any kind or order *algebraically* as in the above example.

I will however give one more example of reference to series for the sake of another fact that it will develop.

Suppose we have the fraction $x \div y$ with the desire to find its differential. Let x equal 1, 2, 3, &c., successively, and y equal 1, 3, 5, 7, &c., then $dx = 1$ and $dy = 2$. Considering them in separate moments as above we have, moment a, b, c, d,

$$\frac{1}{1},\quad \frac{2}{3},\quad \frac{3}{5},\quad \frac{4}{7}, \text{ \&c., \&c.}$$

Take for example the moment b, and the fraction increases, or rather *decreases* from $\frac{10}{15}$ to $\frac{9}{15}$, or $\frac{1}{15}$. If we take the next moment or c it decreases from $\frac{21}{35}$ to $\frac{20}{35}$ or $\frac{1}{35}$. For the next moment the decrease is $\frac{1}{63}$.

Here we observe that the numerator of these amounts of decrease for each successive moment is uniform, -1. If now we take the fraction for any moment as $2 \div 3$ and multiply the differential of the numerator 1 by the denominator for the moment 3, and from their product $= 3$ subtract the product of the numerator by the differential of the denominator $2 \times 2 = 4$, we have -1, which is the constant numerator found above.

Now whenever both the numerator and the denominator of the fraction increase or decrease uniformly—that is, each of them have any constant for their differential coefficient, we should have the same phenomenon, namely, a constant numerator for the amount of the change and this constant is always the differential of the numerator minus the differential of the denominator; that is, when the rate of change is uniform in both the members of the fraction, the usual form $ydx - xdy$ is the same as $dx - dy$, and the coefficients y and x are unnecessary.

If now we turn to consider the danominators obtained above for the successive moments, we shall see that we obtained them by multiplying the value at the beginning of a moment as 3 into the value at the end of that moment as 5—giving us 15, 35, 63, &c. If we are to regard the $-1 \div 15$ as the *amount* of change—this difference is important. But if we are to regard the moment as an indivisible amount of time, and the $-1 \div 15$ as the rate of change *at* that moment rather than the amount

of change *during* any moment or amount of time, the difference between y' and y'' becomes nothing and $y' \times y''$ is the same as y^2. Hence the formula $(ydx - xdy) \div y^2$, and when x and y taken separately change uniformly, we have $(dx - dy) \div y^2$ for the differential of the fraction $x \div y$.

And in all this there is no assumption that the differential is small, or that it may be treated as nothing on one side of an equation while we use it as something on the other; and there is no need of any such assumption. For the sake of convenience the unit or moment of time should be made the same far all the variables that enter into any one equation. But it may be different for each set of variables between which we may have occasion to make an equation.

The method of finding differentials above suggested is based on the principle that "the *amount* of change *during* any moment, considered as a quantity of time, is, when divided by the unit of time, the *rate* of change *at* the moment, considered as an indivisible point of time." Hence we first find the amount of change in a moment in numbers and replace those numbers, factor by factor, with the algebraic symbols representing the variables and the differentials of each of them. We have then a general expression for the rate of change of the compound variable.

It may be said that this is after all only experimental. It is merely a numerical computation, and has not the generality and abstraction of form that mathematicians demand.

Perhaps so. But, then, there is no algebraic method of proving these formulae, that has yet fallen under my notice, that does not encounter some one or more of the difficulties I mentioned at the beginning of this paper. There is no one of them that I know of that does not involve an absurdity, and that may not be shown to be false by what is known as the indirect method of proof as *reductio ad absurdum*.

But again. We must accept some things that cannot be proved algebraically or by general formula. The multiplication table cannot be *proved* in that way, that I know of. I do not think that any man has ever yet proved algebraically, or by the use of letters, that $3 \times 4 = 12$. He may say, "let a represent 3 and b 4, and then the product will be ab." So indeed we may call it, and so we may write it. But who shall prove to us that ab is twelve rather than thirteen or eleven?

There are doubtless cases in which we must make the unit of in-

crease exceedingly small; thus, in the rectification of curves, we can measure the chord of an arc, the arc itself we cannot measure; therefore, if we will express the rate of variation of the arc in units of increment, which are exceedingly small, we may take the length of its chord, which we can measure, for the length of an arc, which we cannot, and we get an approximate value for the circumference of the circle in the terms of its radius. But it seems to me we had better leave the explanation of these cases until they actually occur in the course of the application of the Calculus.

The necessity for so regarding the differential is not found in the nature of the differential at all,—but it is found in the nature of the cases to which we apply the Calculus.

I take the more interest in this matter because of the great importance I attach to the Calculus as a means of mental discipline and culture; and this value does not arise at all from its being a difficult study —one that requires patience, perseverence and concentration of thought. It is rather because it puts the mind into a new attitude in regard to all things, and enables the man of thought to see them in a new light and in new relations. I can hardly regard any one as capable of comprehending the highest, the most general and the most comprehensive truths, without the power and the habit of looking at them from the point of view to whice the study of the Calculus will, of necessity, carry him.

But I believe that the subject has been very unnecessarily involved in metaphysical subtleties—not to say absurdities—and that the difficulty of comprehending it, and the consequent disinclination to study it so common in all our schools and colleges, has arisen chiefly from this unnecessary embarrassment. Adopt the explanation I have now given, treat the differential dx as you treat any other factor, and the Calculus may be made as intelligible as the multiplication table. It may be applied to the simplest operations in arithmetic, or in the proof and solution of the simplest problem in geometry—may, in fact, be understood by all persons. and be none the less powerful and wonderful as a means of science in the attainment of its most recondite facts and laws, on that account.

It will also be observed that the formulae given by my method do not differ at all from those in common use. The only difference is in the

explanation given to them and the method of finding and proving them. Nor will this method, so far as I can see, while making them intelligible and bringing them within the easy comprehension of all persons, put the learner in any position of disadvantage in reference to the higher and more difficult questions that he must encounter in his further pursuit of knowledge or in the practical applications of it to the various purposes of life.

ON THE DIFFERENTIAL EQUATIONS OF DYNAMICS.

BY G. W. HILL.

The general formula of dynamics is

$$\Sigma\left[\left(m\frac{d^2x}{dt^2}-\mathrm{X}\right)\delta x+\left(m\frac{d^2y}{dt^2}-\mathrm{Y}\right)\delta y+\left(m\frac{d^2z}{dt^2}-\mathrm{Z}\right)\delta z\right]=0.$$

In the usual treatment of this equation, we have been asked to attribute to the symbols δx, δy, δz, &c., the signification they have in the calculus of variations. This however is unnecessary, except when we wish to deduce from it the principle of least action; and the student, unacquainted with this calculus, may regard these symbols as multipliers, which, when all the points of the system are free, have any finite values we please, but when the coordinates are restrieted to satisfy an equation $\mathrm{U}=0$, are subject to the condition

$$\frac{d\mathrm{U}}{dx}\delta x+\frac{d\mathrm{U}}{dy}\delta y+\frac{d\mathrm{U}}{dz}\delta z+\&c.=0,$$

an equation which, for brevity, we shall write $\delta\mathrm{U}=0$.

We shall confine our attention to those cases in which the equations of condition and the accelerating forces are functions of the coordinates and the time only, and in which the latter are equivalent to the partial differential coefficients of a single function Ω taken with respect to the coordinates whose acceleration they express.

Whenever a function as U involves, in addition to x, y, z, &c., their first differential coefficients with respect to the time, quantities which we shall denote by x', y', z', &c., we shall suppose that $\delta\mathrm{U}$ involves, besides the terms written above, the following

$$\frac{dU}{dx'}\delta x' + \frac{dU}{dy'}\delta y' + \frac{dU}{dz'}\delta z' + \&c.$$

Moreover as we shall have to differentiate such functions as δU with respect to t, we shall meet with such quantities as $\frac{d\delta x}{dt}$, and shall suppose that the order of the symbols d and δ may be inverted, that is, we shall have equations such as

$$\frac{d\delta x}{dt} = \delta\frac{dx}{dt} = \delta x'.$$

The reader will see in this only a notational assumption, without quantitative significance, serving merely as machinery of demonstration. It will be noted that t is a variable not subject to the operation δ.

We have

$$\Sigma(X\delta x + Y\delta y + Z\delta z) = \delta\Omega,$$

and for convenience may put

$$\tfrac{1}{2}\Sigma m\left(\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2} + \frac{dz^2}{dt^2}\right) = T.$$

Then it will readily be perceived that the general formula can bc written thus

$$\frac{d}{dt}.\Sigma m\left(\frac{dx}{dt}\delta x + \frac{dy}{dt}\delta y + \frac{dz}{dt}\delta z\right) - \delta(T + \Omega) = 0.$$

The coordinates x, y, z, &c., can be expressed as functions of the time and certain variables q_i, independent of each other and whose number is equal to that of the variables x, y, z, &c. diminished by the number of equations of condition. Substituting for x, y, z, &c. theil values in terms of the new variables q_i, it is plain that the last equation will take the following form,

$$\frac{d}{dt}.\Sigma_i p_i\delta q_i - \delta(T + \Omega) = 0.$$

We can find the value of p_i, without actually making the substitution, from this consideration; since the original equation contains only the variations δx, δy, δz, &c. without the variations $\delta\frac{dx}{dt}$, $\delta\frac{dy}{dt}$ $\delta\frac{dz}{dt}$ &c., it fol-

lows that, in its transformed state, it should contain only the variations δq_i without the variations $\delta\frac{dq_i}{dt}$.

Then writing q_i' for $\frac{dq_i}{dt}$, the coefficient of $\delta q_i'$ should vanish in the equation

$$\Sigma_i\left(\frac{dp_i}{dt}\delta q_i + p_i\delta q_i'\right) - \delta(\mathrm{T} + \Omega) = 0.$$

That is, since Ω does not contain q_i',

$$p_i = \frac{d\mathrm{T}}{dq_i'}.$$

Thus the general formula becomes

$$\frac{d}{dt}.\ \Sigma_i\left(\frac{d\mathrm{T}}{dq_i'}\,\delta q_i\ \right) - \delta(\mathrm{T} + \Omega) = 0.$$

Because in this equation the variables q_i are independent, we may equate the coefficient of each δq_i to zero. Thus

$$\frac{d}{dt}.\frac{d\mathrm{T}}{dq_i'} - \frac{d(\mathrm{T} + \Omega)}{dq_i} = 0.$$

This is Lagrange's canonical form of the differential equations of motion.

A simpler form may be obtained by substituting the variables p_i for q_i'. By adding to and subtracting from the general formula, the term $\delta.\Sigma_i(p_iq_i')$, and writing

$$\mathrm{H} = \Sigma_i(p_iq_i') - \mathrm{T} - \Omega,$$

it becomes

$$\Sigma_i\left(\frac{dp_i}{dt}\,\delta q_i - \frac{dq_i}{dt}\,\delta p_i\right) + \delta\mathrm{H} = 0.$$

Equating the coefficients of each variation δq_i and δp_i to zero gives the equations

$$\frac{dp_i}{dt} = -\frac{d\mathrm{H}}{dq_i}, \qquad \frac{dq_i}{dt} = \frac{d\mathrm{H}}{dp_i},$$

which are known as Hamilton's canonical form.

The expression for H can take a simpler shape. From the value of T, it is evident that a certain part of it is independent of the variables q_i,

which may be denoted by T_0, another part T_1, involves the first powers, and a third T_2 involves the squares and products of the same; then $T = T_0 + T_1 + T_2$. By the theory of homogeneous functions

$$\Sigma_i(p_i q_i') = \Sigma_i\left(\frac{dT}{dq_i'} q_i'\right) = T_1 + 2T_2.$$

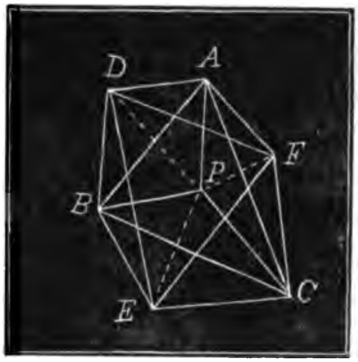
Hence, writing

$$\Omega' = \Omega + T_0,$$
$$H = T_2 - \Omega'.$$

GEOMETRICAL PROBLEM.

BY T. P STOWELL, ROCHESTER, N, Y.

In any triangle whose angles are all known, as also the distances from the three angles to any point either within or without the triangle, the sides may be found as follows:



Let ABC be the triangle, and P a point whose distances PA, PB and PC are given; revolve the triangle APC over the side AC, P will be at F; and APB revolved over AB, P will be at D; also BPC revolved over BC, P will be at E; then join DE, EF and FD.

The $\triangle$ AFD is known, having two equal sides, AF and AD, and the included angle DAF double the angle BAC. Hence DF is known. Similarly FE and DE are known, and therefore all the angles of the $\triangle$ DEF are known. Consequently $\angle$ APC = AFD + DFE + CFE, and therefore the sides AC, AB and BC are easily found. This question admits of an easy geometrical construction.

Note on the Solution of Quadratic Equations, by Wm. Rock, C. E., Eagle Village, N. Y.—The following method of solving Quadratic Equations does not appear in many excellent books on Algebra. The object of the method is to solve any quadratic equation without involving fractions.

$$\text{Let } ax^2 + 2bx = n \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

be any quadratic. Assume $x = y \div a$, then will $ax^2 = y^2 \div a$, and $2bx = 2by \div a$. Substitute these values in the given equation and we have

$$\frac{y^2}{a} + \frac{2by}{a} = n \ldots\ldots\ldots\ldots\ldots\ldots 2).$$

Multiply (2) by a and we have

$$y^2 + 2by = an \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

We have thus removed the coefficient of x^2 without changing that of x. Solving (3) by the ordinary rule, we have

$$y = -b \pm \sqrt{an + b^2}.$$

Hence

$$x = \frac{y}{a} = \frac{-b \pm \sqrt{an + b^2}}{a},$$

which gives the following arithmetical

Rule: If the coefficient of x is an odd number double the whole eqution, then multiply the right hand member by the coefficient of x^2, to the product add the square of half the coefficient of x, extract the square root of the result, connect this square root with half the coefficient of x with its sign changed and divide this result by the coefficient of x^2; the quotient is the value of x.

Numerical Example.—Let $11x^2 - 17x = 108$.

Since the coefficient of x is odd, double the equation and apply the rule to the resulting equation:

$22x^2 - 34x = 216$. Then $216 \times 22 = 4752$, add $(\frac{1}{2} \times 34)^2 = 17^2 = 287$, and we have $4752 + 289 = 5041$, extracting square root of this sum we have $\sqrt{5041} = \pm 71$. Hence

$$x = \frac{17 \pm 71}{22} = 4 \text{ or } - 2\tfrac{5}{11}.$$

SOLUTIONS OF PROBLEMS IN NO. 9.

Solutions of problems in No. 9 have been received as follows:

From A. L. Baker, 38 & 40; Geo. M. Day, 38 & 39; Prof. A. B. Evans, 37, 38, 39 & 40; Henry Gunder, 38 & 39; Henry Heaton, 38, 39 & 40; Artemas Martin, 37 & 38; Walter Siverly, 38 & 39; E. B. Seitz, 38 & 40. S. W. Salmon and R. M. DeFrance each sent an elegant solution to Werner Stille's quest in Curves of Pursuit.

37.—"Divide unity into three parts such that if each part be increased by unity the sums shall be three rational cubes."

SOLUTION BY ARTEMAS MARTIN, ERIE, PA.

Let $a^3x^3 - 1$, $b^3x^3 - 1$ and $c^3x^3 - 1$ be the parts required.

Then $a^3x^3 - 1 + b^3x^3 - 1 + c^3x^3 - 1 = 1$,

or $$(a^3 + b^3 + c^3)x^3 = 4 = A \qquad (1).$$

$$\therefore \quad x^3 = \frac{A}{a^3 + b^3 + c^3} \qquad (2).$$

Let $a^3 + b^3 = t$, then $x^3 = \dfrac{A}{t + c^3}$ (3).

Let $p - s = c$. Then, putting $r^3A = t - s^3$,

$$x^3 = \frac{A}{p^3 - 3p^2s + 3ps^2 + r^3A} = \frac{1}{\frac{p^3}{A} - \frac{3p^2s}{A} + \frac{3ps^2}{A} + r^3}.$$

Assume $$\frac{p^3}{A} - \frac{3p^2s}{A} + \frac{3ps^2}{A} + r^3 = \left(r + \frac{ps^2}{r^2A}\right)^3;$$

by reduction, $$p = \frac{3r^3As}{r^3A - s^3}.$$

$$\therefore \; c = s\left(\frac{2r^3A + s^3}{r^3A - s^3}\right), \; x = \frac{r^2A}{r^3A + ps^2} = \frac{1}{r}\left(\frac{r^3A - s^3}{r^3A + 2s^3}\right),$$

and $\dfrac{a^3}{r^3}\left(\dfrac{r^3A - s^3}{r^3A + 2s^3}\right)^3 - 1$, $\dfrac{b^3}{r^3}\left(\dfrac{r^3A - s_8}{r^3A + 2s^3}\right)^3 - 1$, $\dfrac{s^3}{r^3}\left(\dfrac{2r^3A + s^3}{r^3A + 2s^3}\right)^3 - 1$,

are the parts required.

We must now solve the equation $a^3 + b^3 = t$. $y^3 + z^3$
$= (y+z)(y^2 - yz + z^2)$.

Suppose $y^2 - yz + z^2 = m^3$, and y and z prime to each other; then $\left(\frac{y}{m}\right)^3 + \left(\frac{z}{m}\right)^3 = y + z$, and in general

$$\left(\frac{y-(n+1)z}{m}\right)^3 + \left(\frac{(n+1)y - nz}{m}\right)^3 = (n^2+n+1)[(n+2)y - (2n+1)z].$$

When $n = 0$, $$\left(\frac{y-z}{m}\right)^3 + \left(\frac{y}{m}\right)^3 = 2y - z \ldots\ldots\ldots\ldots (4).$$

Hence we may take $a = \frac{y-z}{m}$, $b = \frac{y}{m}$, and t $2y - z$.

Put $y = v + w$, $z = v - w$; then $y^2 - yz + z^2 = v^2 + 3w^2 = m^3$, which is satisfied by

$$v = p(p+3q)(p-3q),\ w = 3q(p+q)(p-q),\ m = p^2 + 3q^2;$$
$$\therefore y = p(p+3q)(p-3q) + 3q(p+q)(p-q),$$
$$z = p(p+3q)(p-3q) - 3q(p+q)(p-q).$$

Take $p = 11$, $q = 2$; then $y = 1637$, $z = 233$, $m = 133$ and $t = 3041$; $a = \frac{1404}{133}$, $b = \frac{1637}{133}$.

But we must have $\quad r^3A + s^3 = 4r^3 + s^3 = t = 3041.$

Let $r = 9$, then $2916 + s^3 = 3041$, $s^3 = 125$, and $s = 5$. $\therefore x = \frac{2791}{28494}$, and the parts are

$$\left(\frac{3918564}{3789702}\right)^3 - 1 = \frac{5743015291812773736}{5442709850427501 6408},$$

$$\left(\frac{3961405}{3789702}\right)^3 - 1 = \frac{7738158893915488717}{54427098504275016408},$$

$$\left(\frac{4568867}{3789702}\right)^3 - 1 = \frac{40945924318546753955}{54427098504275016408}.$$

See *Mathematical Miscellany*, pp. 118–123, whence the substance of this solution is taken.

38.—"Prove that $\frac{1}{\underline{|2}} - \frac{1}{\underline{|3}} + \frac{1}{\underline{|4}} - \ldots\ldots \frac{(-1)^x}{\underline{|x}} = \frac{1}{e}$; where e is the base of Napierian logarithms."

SOLUTION BY E. B. SEITZ, GREENTILLE, OHIO.

By a well known theorem,

$e^x = 1 + \frac{x}{1} + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \text{\&c.}$ Making $x = -1$, we have

$\frac{1}{\underline{|2}} - \frac{1}{\underline{|3}} + \frac{1}{\underline{|4}} - \ldots\ldots\ldots\ldots \frac{(-1)^x}{\underline{|x}} = e^{-1} = \frac{1}{e}$. Q.E.D.

39.—"Two weights are connected by a fine string which passes over a pulley; if the weights be 50 and 72 lbs., determine what stationary weight the string must be able to support, that it may just escape breaking during the motion."

SOLUTION BY HENRY GUNDER, GREENVILLE, OHIO.

As illustrated by Attwood's Machine, we have a moving force $= 22g$, and a mass to be moved $= 122$; hence the velocity acquired in one second is $\frac{22}{122}g = \frac{11}{61}g$. The 72 pound weight will have acquired this velocity in one second, but moving freely under the influence of gravity its velocity would have been g. $\therefore$ there must have been an opposing force of $\frac{50}{61}g$, or $\frac{50}{61}$ of 72, on the string $= 59\frac{1}{61}$ lbs.

40.—(See ANALYST No. 9.)

SOLUTION BY PROFESSOR ASHER B. EVANS, LOCKPORT, N. Y.

We will take for the axis of y that side of the canal along which A walks and for the axis of x a line perpendicular thereto at an arbitrary point, the canal being on A's right hand side and in the direction of x positive.

Let the time t be computed from an arbitrary instant, and let x' and y' be the coordinates of B when A is on the axis of y at a distance from the origin expressed by $b \div nt$, b being an arbitrary constant depending on the situation of the origin and the instant at which the time t is supposed to commence. Then the right line joining B and A will have for its equation $x - x' = \frac{x'}{y' - b - nt}(y - y'$ ....(1); from which it follows that this line will make with the axes of x and y angles whose

cosines are $\dfrac{x'}{\sqrt{x'^2 + (y' - b - nt)^2}}$ and $\dfrac{y' - b - nt}{\sqrt{x'^2 + (y' - b - nt)^2}}$. Two constant forces are acting upon B, the force r due to his rowing and acting in the direction BA, and the force m due to the current and acting in the direction opposite to y positive. The components of the force r in the direction of x and y are $\dfrac{rx'}{\sqrt{x'^2 + (y' - b - nt)^2}}$ and $\dfrac{r(y' - b - nt)}{\sqrt{x'^2 + (y' - b - nt)^2}}$. Considering that the former of these compo- nents constantly tends to diminish the coordinate x', and that the latter tends to diminish y' when $y' > (b + nt)$ and to increase it when $y' < (b + nt)$, for the motion of B we shall have, from well known principles of mechanics, on omitting the accents as henceforth useless,

$$\frac{dx}{dt} = -\frac{rx}{\sqrt{x^2 + (y - b - nt)^2}} \text{ and } \frac{dy}{dt} = -\frac{r(y - b - nt)}{\sqrt{x^2 + (y - b - nt)^2}} - m. \quad (2).$$

To integrate equation (2) put $y - b - nt = x\tan\psi$ ....(3);

$$\text{then } \frac{dx}{dt} = -r\cos\psi \text{ and } \frac{dy}{dt} = -r\sin\psi - m \quad \text{............(4).}$$

Differentiating (3) and dividing by dt, we obtain

$$\frac{dy}{dt} - n = \frac{dx}{dt}(\tan\psi) + \frac{x}{dt}d(\tan\psi) \quad \text{............(5).}$$

Substituting in (5) the values of $\frac{dx}{dt}$ and $\frac{dy}{dt}$ given by (4),

$$m + n = -\frac{x}{dt}d(\tan\psi) \quad \text{.........................(6).}$$

$$\text{But } \frac{dx}{dt} = -r\cos\psi = -\frac{r}{\sqrt{1 + \tan^2\psi}} \quad \text{..........(7),}$$

and from (6) and (7) on putting $\dfrac{m + n}{r} = p$ we find

$$p\frac{dx}{x} = \frac{d(\tan\psi)}{\sqrt{1 + \tan^2\psi}} \quad \text{.........................(8).}$$

Integrating (8) $\log\left(\frac{x}{c'}\right)^p = \log\left\{\tan\psi + \sqrt{1 + \tan^2\psi}\right\}$, or better

$$\left(\frac{x}{c'}\right)^p = \tan\psi + \sqrt{1 + \tan^2\psi} \dots\dots\dots\dots\dots (9),$$

where c' is an arbitrary constant. To determine this constant, we observe that at a certain time the line joining B and A is perpendicular to the side of the canal, or to the axis of y. Taking the origin, which we have not hitherto completely fixed, at the point where this perpendicular meets the axis of y, denoting the width of the canal by c, and taking the instant when A is at the origin of coordinates for the origin of t, we have when $t = 0$, $b = 0$, $y = 0$ and $x = c$; whence from equation (3) $\tan\psi = 0$, and then from equation (9) $\left(\frac{x}{c'}\right)^p = \left(\frac{c}{c'}\right)^p = 1$ and $c' = c$ the width of the canal.

Equation (9) now gives $\tan\psi = \frac{1}{2}\left\{\left(\frac{x}{c}\right)^p - \left(\frac{x}{c}\right)^{-p}\right\}$; and therefore

$$\sin\psi = \frac{\left(\frac{x}{c}\right)^p - \left(\frac{x}{c}\right)^{-p}}{\left(\frac{x}{c}\right)^p + \left(\frac{x}{c}\right)^{-p}},\ \cos\psi = \frac{2}{\left(\frac{x}{c}\right)^p + \left(\frac{x}{c}\right)^{-p}} \dots\dots (10).$$

Eliminating dt from equation (4)

$$\frac{dy}{dx} = \frac{r\sin\psi + m}{r\cos\psi} = \left(\frac{r + m}{2r}\right)\left(\frac{x}{c}\right)^p - \left(\frac{r - m}{2r}\right)\left(\frac{x}{c}\right)^{-p} \dots\dots (11).$$

Integrating equation (11), remembering that $x = c$ when $y = 0$, we have

$$\frac{2ry}{c} = \frac{r + m}{p + 1}\left\{\left(\frac{x}{c}\right)^{p+1} - 1\right\} + \frac{r - m}{p - 1}\left\{\left(\frac{x}{c}\right)^{1-p} - 1\right\} \dots (12).$$

When $y = a$, $x = 0$ and equation (12) gives $a = \frac{cnr}{[r^2 - (m+n)^2]}$; and therefore $c = \frac{a}{nr}[r^2 - (m + n)^2] =$ the width of the canal.

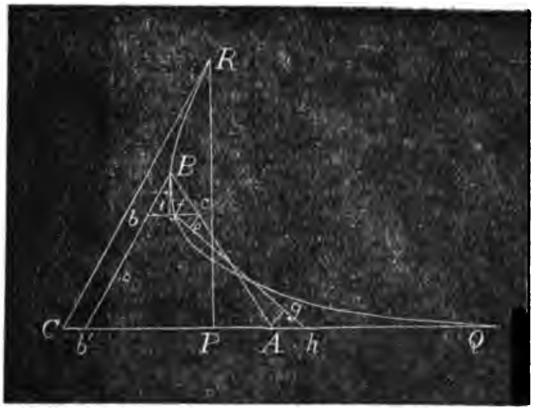
Note.—The foregoing general solution includes the problem proposed by Werner Stille on p. 145 of The Analyst. For putting $m = 0$, $r = n^2$ and $c =$ the line AB, we have from equation (12)

$$y = \frac{c}{2}\left[\frac{n}{n + 1}\left\{\left(\frac{x}{c}\right)^{\frac{n+1}{n}} - 1\right\} - \frac{n}{n - 1}\left\{\left(\frac{x}{e}\right)^{\frac{n-1}{n}} - 1\right\}\right]$$

for the equation of the curve pursued by the dog.

[Messrs. A. L. Baker, of Lafayette College, Easton, Pa., and H. Heaton, of Des Moines Public Schools, solved No. 40 by determining the Eq. to the curve which represents the *trace* of the boat in the water, in which case the question may be solved exactly as Mr. Stille's question, by substituting $m + n$ for the velocity of the man ($= 1$ in Mr. Stille's quest.) and r for n, the velocity of the dog.

The foregoing elegant solution of No. 40 is all that could be desired as a solution to questions of this kind. We add, however, the following solution based on geometrical considerations, hoping that it may assist the student in comprehending the solution of these questions:



Put the required distance PR $= p$, and produce QP to C, making PC : PQ :: $m : n$, then is PC $= ma \div n = a'$.

Let A and B represent contemporaneous positions of the two men at the end of any time t. Join RC, and draw Bb parallel to RC. Take a point c on the line AB infinitely near B, and draw $b'c$ parallel to CQ. Put C$b = x$ and AB $= y$.

As the distance traversed by the boat from B towards A, if not influenced by the current, would be proportional to the time, we shall have B$c = rdt$; but during the time dt the current will carry the boat in the direction of the line cb' a dittance $cf = mdt$, the resultant of the two forces being the curve Bf; $\therefore b'f = dx$. Draw Bi perpendicular to $b'c$, then is $b'i = mdt$, $\therefore ci = b'f = dx$, and $fi = dx - mdt$.

If while B describes the curve Bf, A advances to h, we shall have Ah $= ndt$. Join fh and draw Ag and ce each perpendicular to fh. Then by similar triangles we have, B$c : ci :: cf : ef$, or $rdt : dx :: mdt : mdx \div r = ef$. Also B$c : ci ::$ A$h : gh$, or $rdt : dx :: ndt : ndx \div r = gh$.

Now the decrement of AB is obviously $ef + gh -$ Bc, therefore we have, $dy = \frac{m}{r} dx + \frac{n}{r} dx - rdt = \frac{m + n}{r} dx - rdt$. Putting $(m + n) \div r = q$ and integrating we have

$$y = p + qx - rt \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

p being the constant introduced by integration.

Because $bA = a' + nt - x$, we have, by similar triangles, $y : a' + nt - x :: rdt : mdt + dx$. Substituting for y from (1) and reducing we get

$$\frac{dt}{dx} = \frac{p + qx - rt}{(a'r - mp) - (r + mq)x + r(m + n)t} = \frac{p + qx - rt}{p' - q'x + r't} \ldots\ldots(2).$$

When $y = 0$, in (1), $x = a + a'$, and $rt = ra \div n$.

$$\therefore\ p = \frac{ra}{n} - \frac{m + n}{r}.(a + a') = \frac{ra}{n} - \frac{m + n}{r}.a - \frac{m + n}{r}.\frac{m}{n}.a$$

$$= \frac{r^2a}{rn} - \frac{mn + n^2}{rn}.a - \frac{m^2 + mn}{rn}.a = \frac{a[r^2 - (m + n)^2]}{rn}.$$

If it were required to determine the length, z, of the curve, because $dz^2 = Bf^2 = Bi^2 + if^2 = Bc^2 - ci^2 + (ci - cf)^2 = r^2dt^2 - dx^2 + dx^2 - 2mdxdt + m^2dt^2 = r^2dt^2 + m^2dt^2 - 2mdxdt$,

$$\therefore\ dz = \sqrt{(r^2 + m^2)dt^2 - 2mdxdt} \ldots\ldots\ldots(3).$$

Because (2) is reducible to a homogeneous equation it may be integrated in finite terms, and hence, by substitution in (3) we have dz in functions of a single variable, which being integrated will give the length of the curve.

Messrs. S. W. Salmon, of Mt. Olive, N. J., and R. M. DeFrance, of Mercer, Pa., each sent an elegant solution of Mr. Stille's question, but as either of the above methods applies equally well to that case we reluctantly omit their publication.—Ed.]

PROBLEMS.

41. By J. P. Child, Salem, Iowa.—Given $x^2 + y = 7 \ldots\ldots\ldots(1)$,

$$x + y^2 = 11 \ldots\ldots\ldots\ldots\ldots(2),$$

to find the values of x and y.

42. By Philip Hoglan, Newcomerstown, Ohio.—Find two integral numbers the difference of whose squares is a cube, and the difference of their cubes a square.

43. By L. Regan, Boonsboro, Iowa.—Let AB and AC be two lines intersecting each other at right angles in A, and D, any point given in position. Required the position of a line EF through D intersecting the lines AB and AC in E and F, so that $(ED)^2 + (DF)^2 = m^2$.

44. By George L. Dake, Cleveland, Ohio.—If a circle be divided into three equal parts by two parallel chords, find the perpendicular distance between the chords in terms of the radius.

45. By Elias Schneider, A. M., Sunbury, Pa.—Required the area and sides of an obtuse angled triangle whose angles are to one another as 2, 3 and 7 and whose longest side equals 1. No logarithms to be used in the solution.

46. By Kate R. Mason.—The area of the piston of a high-pressure engine is 1200 square inches, the length of stroke $8\frac{1}{2}$ ft., the pressure of steam upon the piston 32 lbs. per sq. inch and the number of strokes per min. 18. Required the number of cubic feet of water the engine will raise from a mine 60 fath. deep, the friction being estimated at 1 lb. per sq. inch, plus the pressure of the atmosphere.

47. By Henry A. Roland, Troy, N. Y.—What are the forces of inertia acting on a given point of mass m in the axis of the connecting rod of a steam engine whose crank arm moves with uniform angular velocity w?

48. By Artemas Martin, Erie, Pa.—A cylindrical tower, radius r, is surrounded by a walk, width a. Two persons are on the walk; what is the probability that they can see each other?

49. By Prof. D. M. Sensenig, Millersville, Pa.—How far will a man travel in unwinding an inch rope from a frustrum of a cone whose upper diameter is 2 ft., lower diameter 15 ft., and height 35 ft., the rope to be closely wound around the frustrum from top to bottom?

50. By Prof. A. Hall, Naval Observatory, Washington, D. C.—Assuming the earth's orbit to be a circle, if a comet move in a parabola around the sun and in the plane of the earth's orbit, show that the comet cannot remain within the earth's orbit longer than 78 days.

BOOK NOTICES.

A Complete Algebra, for the use of Schools, Academies and Colleges, by JOSEPH FICKLIN, PH. D., Professor of Mathematics in the University of the State of Missouri; Ivison, Blakeman, Taylor & Co., New York and Chicago, 1874.

From the partial examination we have been able to give this book, and from our knowledge of the ability and accuracy of the author, we have no hesitation in bestowing upon it our highest commendation, as a systematic and thorough treatise on Algebra.

A Treatise on the Principles and Application of Analytic Geometry, by HENRY T. EDDY, C. E., PH. D., Professor of Mathematics and Astronomy in the University of Cincinnati; Philadelphia, Cowperthwait & Co., 1874.

In the production of this treatise, we think Prof. Eddy has rendered an important service to students. Our limited space will not permit us to mention the various excellencies that are manifest on even a cursory examination of the book. We must content ourselves therefore, for the present, in quoting a single paragraph from the preface:—"The form of notation adopted is thoroughly systematized and prepares the student to read with ease the great modern writers upon analytic geometry. The marked value of the angular notation used is a sufficient recommendation for its adoption. For it I am happy to acknowledge my indebtedness to Prof. J. M. Peirce, of Harvard University, from whose book it is borrowed."

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Errata. On page 75, fourth line from bottom, insert the coefficient 2 before "sin $\frac{1}{2}C$" &c., in both numerator and denominator.

On page 157, line 17, for "Put $A_r = x^r = [(x+1)-1]^r = (x+1)_r$," read, Because $x^r = [(x+1)-1]^r = (x+1)^r$, &c.; and on same page, line 19, for "Then" read Therefore.

On page 158, lines 8 and 9 from bottom should read

$$t_1 = \frac{1}{12\pi}\left[\left(2+c\right)^{\frac{3}{2}} + \left(2-c\right)^{\frac{3}{2}}\right],$$
$$t_2 = \frac{1}{12\pi}\left[\left(2+c\right)^{\frac{3}{2}} - \left(2-c\right)^{\frac{3}{2}}\right].$$

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ON THE MAXIMA AND MINIMA OF ALGEBRAIC POLYNOMIALS.

BY PROF. DAVID TROWBRIDGE, WATERBURGH, N. Y.

1. Let $u = A_1x^n + A_2x^{n-1} + A_3x^{n-2} + \ldots + A_{n-1} = \psi x \ldots\ldots(1)$
The expression ψx is read "function of x".

Suppose we wish to find a value of x such that if we substitute it in the polynomial (1), this polynomial, or u, will be a maximum, or the greatest possible; or a minimum, or the least possible, according to the conditions of the problem. To discover such values of x,—for there may be several maxima or minima of different orders,—we shall proceed as follows:

Suppose x to be increased or decreased by a small quantity h, so that we have, if we accent the u for this case,

$$u' = \psi(x+h), \text{ or } u' = \psi(x-h) \ldots\ldots\ldots (2)$$

Let each of these be developed according to the ascending powers of h, and let the result be denoted as follows:

$$u' = \psi x + \psi' x h + \psi'' x \cdot \frac{h^2}{1.2} + \psi''' x \cdot \frac{h^3}{1.2.3} + \ldots \quad (3)$$

$$u' = \psi x - \psi' x h + \psi'' x \cdot \frac{h^2}{1.2} - \psi''' x \cdot \frac{h^3}{1.2.3} + \ldots \quad (4)$$

In these equations $u = \psi x$, and if we transpose this term we shall have

$$u' - u = \psi' x.h + \psi'' x.\frac{h^2}{1.2} + \psi''' x \cdot \frac{h^3}{1.2.3} + \ldots \quad (5)$$

$$u' - u = -\psi' x.h + \psi'' x.\frac{h^2}{1.2} - \psi''' x \cdot \frac{h^3}{1.2.3} + \ldots \quad (6)$$

In these equations $\psi' x$, $\psi'' x$, $\psi''' x$, &c., are finite quantities; and if we take h very small, $\psi' x.h$ will be of the 1st order of small quantities, or infinitesimals, and $\psi'' x \times (h^2 \div 1.2)$, of the 2nd order, &c., so that $\psi'' x \times (h^2 \div 1.2)$ is infinitely small as compared with $\psi' x.h$, and $\psi''' x \times (h^3 \div 1.2.3)$ is infinitely small as compared with $\psi'' x \times (h^2 \div 1.2)$, &c. (See Prof. Ficklin's *Complete Algebra*, p. 129.) Now if u, or its equal the polynomial,

has its greatest value, or is a maximum, then u', which is found by increasing or diminishing x by a quantity h, no matter how small, is less than u; and if u is the least possible, or a minimum, then u' is greater than u. We hence see that the infinitesimal value of h, which we have supposed, is all that we need consider in this demonstration. We can easily see, then, that when h is thus small, the term $\phi'x.h$ will be *greater* than *all* the other terms; for the number of terms in the second members of (5) and (6), is limited, so that their number cannot compensate for their smallness. In the case of a maximum $u'-u$ is negative; and in the case of a minimum $u'-u$ is positive. But because $\phi'x.h$ is greater than all the other terms in the second members of (5) and (6), the *sign* of this term will determine the *sign* of the second member; and from (5) we see that when h is positive, supposing $\phi'x$ positive, the second member is positive, and hence $u'-u$ is positive for a maximum; and from (6) we see that $u'-u$ is negative if h is negative; that is, the second member, or $u'-u$ changes sign with h. But if u is a maximum, whether x be increased or diminished, $u'-u$ is *negative*. To satisfy all these conditions we must make $\phi'x=0$. For a minimum we must have in all cases, $u'-u$ a *positive* quantity, and in order to secure this, we must also have $\phi'x=0$.

The term $\phi''x \times (h^2 \div 1.2)$, which is greater than all the remaining terms, has the same sign whether h be plus or minus. For a maximum we see further, then, that $\phi''x$ must be *negative*; and for a minimum, $\phi''x$ must be *positive*.

Our conditions, then, are, for a maximum,

$$\phi'x=0, \text{ and } \phi''x \text{ negative, and}$$
$$\phi'x=0, \text{ and } \phi''x \text{ positive,}$$

for a minimum.

Now $\phi'x$ is called the first derived polynomial, or function, and $\phi''x$, the second derived polynomial, &c. (See Ficklin's *Com. Algebra*, p. 400, where he uses fx instead of ϕx.) To find what value of x will render u a maximum, put the first derived polynomial equal to 0, and find the value of x from the resulting equation. Substitute these values, in succession, in the second derived polynomial, and all the values that give a *negative* result indicate so many maxima; and all that give a *positive* result, indicate so many minima. If any of the values of x render $\phi''x=0$, then, in order that such values of x may produce maximum or minimum values of u, we must also have $\phi'''x=0$, for $\phi'''x \times (h^3 \div 1.2.3)$ changes sign with h; and to determine whether we have a maximum or a minimum, we must substitute the values of x in $\phi''''x$, which will be negative for a maximum and positive

for a minimum. For other cases we proceed in a similar manner with the other terms.

This demonstration will apply to any function of x from which we can find $\phi'x$, $\phi''x$, &c. If the student masters this demonstration, he will find it of great service to him when he takes up the Calculus, which at once furnishes the means of finding $\phi'x$, $\phi''x$, &c.

Illustrative problem:—Divide a in to two such parts that the square of the first multiplied by the second shall be a maximum.

Let x be the first part, then $a - x$ is the second, and $u = x^2(a-x) = ax^2 - x^3 = \phi x$ is to be a maximum. In this case $\phi'x = 2ax - 3x^2 = 0$, $\phi''x = 2a - 6x$.

$2ax - 3x^2 = x(2a - 3x) = 0$; $\therefore x = 0$, and $2a - 3x = 0$, $x = \frac{2}{3}a$. If we put the value $x = 0$, in $\phi''x$, we have $2a$ a positive quantity; and if we put the value $x = \frac{2}{3}a$ in $\phi''x$ we have $\phi''x = 2a - 4a = -2a$, a negative quantity. Therefore, the first is a minimum, and the second a maximum.

2. The student will see that the difficulty consists mainly in finding $\phi'x$ and $\phi''x$, and if we could readily find these we could solve many problems in maxima and minima without the Calculus. In many cases it is unnecessary to find $\phi''x$, since we can easily satisfy ourselves whether we have a maximum or a minimum. The following process, which I think the student of algebra can easily understand, will enable us to find both $\phi'x$ and $\phi''x$, and more terms, if we choose, in algebraic functions, or quantities.

If ϕx be an algebraic function of x, then by writing $x + h$ for x, and developing the resulting quantity according to the ascending or positive powers of h, as far as the first and the second power of h, the coefficient of h will be $\phi'x$ (Eqs. (3) and (4)), and that of h^2 will be $\phi''x$. This development can be effected by the Binomial Theorem, or by Indeterminate Coefficients.

Suppose we have

$$u = \phi x = x\sqrt{(1-x^2)} + \frac{\sqrt{(1-x)}}{x}$$

and we wish to find a value of x that will render u a maximum or a minimum. Put $x = x + h$, then

$$u' = (x+h)\sqrt{(1 - x^2 - 2xh - h^2)} + \sqrt{(1 - x - h)}\,(x+h)^{-1}$$
$$= u + \phi'xh \text{ to the first power of } h. \text{ Then}$$

$$u' = x\left(1 + \frac{h}{x}\right)\sqrt{(1-x^2)}\sqrt{\left(1 - \frac{2xh + h^2}{1-x^2}\right)}$$
$$+ \frac{1}{x}\sqrt{(1-x)}\sqrt{\left(1 - \frac{h}{1-x}\right)}\left(1 + \frac{h}{x}\right)^{-1}$$

$$= x\sqrt{(1-x^2)}\left(1+\frac{h}{x}-\frac{xh}{1-x^2}\right)+\frac{1}{x}\cdot\sqrt{(1-x^2)}\left(1-\frac{h}{2(1-x)}-\frac{h}{x}\right)$$

$$= u + h\left(\sqrt{(1-x^2)}-\frac{x^2}{\sqrt{(1-x^2)}}-\frac{1}{2x\sqrt{(1-x)}}-\frac{\sqrt{(1-x)}}{x^2}\right).$$

We must now put the coefficient of h equal to 0, and we shall have

$$\sqrt{(1-x^2)}-\frac{x}{\sqrt{(1-x^2)}}-\frac{1}{2x\sqrt{(1-x)}}-\frac{\sqrt{(1-x)}}{x^2}=0.$$

The solution of this equation will make known the required values of x Take the equation $u = xe^{-x}$, then

$$u' = (x+h)e^{-x+h} = (x+h)e^{-x}.e^{-h} = e^{-x}(x+h)(1-h+\tfrac{1}{2}h^2)$$
$$= e^{-x}(x - hx + \tfrac{1}{2}h^2x + h - h^2) = u + \psi'x.h + \tfrac{1}{2}\psi''x.h^2.$$
$$\therefore e^{-x}(1-x) = 0,\ x = 1.$$

Since $\psi''x = e^{-x}(\frac{1}{2}x - 1)$, the value of $x = 1$, gives $\psi''x = e^{-1}(\frac{1}{2}-1)$ $= -\frac{1}{2}e^{-1}$, a negative quantity, so that this value of x makes $u = e^{-1}$, a maximum.

We could easily extend this demonstration to the case of two independent variables.

ON THE SOLUTION OF CUBIC AND BIQUADRATIC EQUATIONS.

BY G. W. HILL.

In nearly all treatises on algebra the solution of these equations is presented as accomplished by the aid of analytical artifices, which one seems, by some happy hazard, to have stumbled upon. No doubt the processes were found in this manner by the original discoverers, Tartaglia, Carden and Ferrari. But, for many reasons, it would be better to treat the subject as one demanding invention rather than artifice. The equations can, as it were, be interrogated and compelled to yield up their secrets, if they have any.

To say that an equation is solvable algebraically, is to say that an algebraic expression can be found equivalent to the general root, that is, one involving a finite number of the operations of addition, subtraction, multiplication, division and thc extraction of roots of prime degree. If the expression does not involve the last mentioned operation, it is called rational, and if free from the two last, integral.

However complex an algebraic expression involving radicals may be, it it is evident that there must be at least one radical which is involved in it rationally. Supposing this to be denoted by $R^{\frac{1}{n}}$, n being a prime integer, it is not difficult to convince one's self that, by the proper reductions, the expression can be exhibited thus

$$p_0 + p_1 R^{\frac{1}{n}} + p_2 R^{\frac{2}{n}} + \ldots p_{n-1} R^{\frac{n-1}{n}},$$

where p_0, p_1, &c., do not involve the radical $R^{\frac{1}{n}}$. With no loss of generality we can suppose $p_1 = 1$; for if p_1 is not zero, we can multiply the quantity under the radical sign by p_1^n, and then take $(p_1^n R)^{\frac{1}{n}}$ as the radical; and in in the contrary case, if p_k is one of the quantities p which is not zero, the simplification can be accomplished by putting $R' = p_k^n R^k$. Then

$$p_0 + R^{\frac{1}{n}} + p_2 R^{\frac{2}{n}} + \ldots + p_{n-1} R^{\frac{n-1}{n}}$$

may be regarded as the most general form of an algebraic expression.

Here may be enunciated a general proposition, which, although I am not aware that it has ever been proved, is doubtless true, and may be used for purposes of discovery. If an algebraic expression exists, equivalent to the general root of the equation

$$x^m + ax^{m-1} + bx^{m-2} + \ldots + g = 0,$$

it can be exhibited in the above form, n being equal to one of the prime factors of m. Thus the algebraic expression of the root of the general equation of the 5th degree, if it existed, could be presented in the form

$$p_0 + R^{\frac{1}{5}} + p_2 R^{\frac{2}{5}} + p_3 R^{\frac{3}{5}} + p_4 R^{\frac{4}{5}},$$

and that of the 6th degree in either of the two forms

$$p_0 + R^{\frac{1}{3}} + p_2 R^{\frac{2}{3}}, \qquad p_0 + R^{\frac{1}{2}}.$$

Solution of Cubic Equations.

According to the foregoing proposition, the root of the general cubic equation

$$x^3 + ax^2 + bx + c = 0,$$

if it has an algebraic expression, must be presented in the form

$$x = p + R^{\frac{1}{3}} + p' R^{\frac{2}{3}}.$$

But since we suppose that this is an irreducible expression involving radicals, it follows that it must satisfy the given equation, whichever of its three

values is attributed to the radical $\sqrt[3]{R}$. Thus calling either of the imaginary cube roots of unity a, the three roots of the cubic equation must be

$$x_1 = p + R^{\frac{1}{3}} + p'R^{\frac{2}{3}},$$
$$x_2 = p + a\,R^{\frac{1}{3}} + a^2p'R^{\frac{2}{3}},$$
$$x_3 = p + a^2R^{\frac{1}{3}} + a^4p'R^{\frac{2}{3}}.$$

The first method that suggests itself for obtaining equations which shall give the values of p, p' and R, is to substitute these expressions in the symmetrical functions which are equivalent to the several coefficients a, b, c, viz.,

$$x_1 + x_2 + x_3 = -a, \quad x_1x_2 + x_1x_3 + x_2x_3 = b, \quad x_1x_2x_3 = c.$$

But a simpler proceeding is to employ the three symmetrical functions $\Sigma.x$, $\Sigma.x^2$ and $\Sigma.x^3$. Since any cube root, as $\sqrt[3]{R}$, is a root of $x^3 - R = 0$, in which the coefficients denoted above by a and b are each zero, it follows that the sum of the three cube roots of any quantity, as well as the sum of their squares, is zero. Now it is plain that if the value of x is raised to the n^{th} power,

$$x^n = A + BR^{\frac{1}{3}} + CR^{\frac{2}{3}},$$

where A, B and C are free from the radical $\sqrt[3]{R}$, and are consequently the same whichever of the three roots x denotes. Thus since $\Sigma.\sqrt[3]{R} = 0$, $\Sigma.\sqrt[3]{R^2} = 0$,

$$\Sigma.x^n = 3A.$$

Thus, for computing the value of $\Sigma.x^n$, we need only the part A which is free from the radical $\sqrt[3]{R}$. In this way we obtain and equate to their known values in terms of the coefficients a, b, c,

$$\Sigma.x = 3p = -a,$$
$$\Sigma.x^2 = 3(p^2 + 2p'R) = a^2 - 2b,$$
$$\Sigma.x^3 = 3(p^3 + R + 6pp'R + p'^3R^2) = -a^3 + 3ab - 3c.$$

These equations afford the values of p, p' and R; from the first two

$$p = -\frac{a}{3}, \qquad p'R = \frac{a^2 - 3b}{9},$$

and by substitution of these values in the last,

$$R^2 + \frac{2a^3 - 9ab + 27c}{27}R + \left(\frac{a^2\ 3 + b}{9}\right)^3 = 0,$$

a quadratic equation in R; thus the general cubic admits of solution by radicals.

For the sake of brevity putting

$$A = \frac{a^2 - 3b}{9} \qquad B = -\frac{2a^3 - 9ab + 27c}{54},$$

we have

$$R = B \pm \sqrt{(B^2 - A^3)},$$

and as we may take at our option either of the two roots, we have choice of the two expressions for x,

$$x = -\tfrac{1}{3}a + [B + \sqrt{(B^2 - A^3)}]^{\frac{1}{3}} + A\,[B + \sqrt{(B^2 - A^3)}]^{-\frac{1}{3}}$$

$$x = -\tfrac{1}{3}a + [B - \sqrt{(B^2 - A^3)}]^{\frac{1}{3}} + A\,[B - \sqrt{(B^2 - A^3)}]^{-\frac{1}{3}}$$

The three values of x are obtained by attributing in succession to the single cube root appearing in either of these expressions its three values.

I do not know why almost all algebraists prefer to put the root iu the form

$$x = -\tfrac{1}{3}a + \sqrt[3]{[B + \sqrt{(B^2 - A^3)}]} + \sqrt[3]{[B - \sqrt{(B^2 - A^3)}]}.$$

It is certainly easier in practice to make a division than an extraction of a cube root; moreover we are troubled, in the last form, with the selection of the proper three values out of the nine of which it is susceptible, a difficulty which does not occur in the two former expressions.

Solution of Biquadratic Equations.

An algebraic expression for the root of the general equation of the fourth degree,

$$x^4 + ax^3 + bx^2 + cx + d = 0,$$

if it exists, can be presented in the form $P + \sqrt{Q}$. And if this denotes one of the roots, another will be $P - \sqrt{Q}$; but since x has four values, it is plain that P and Q must receive each two values. This condition will be fulfiled if we suppose that these quantities, in their turn, similarly to x, are rational functions of a second radical $\sqrt{R}$. Thus we put

$$P = p + \sqrt{R},$$
$$Q = q + q'\sqrt{R}.$$

Then we have

$$x = p + \sqrt{R} + \sqrt{(q + q'\sqrt{R})}.$$

The four values of x are obtained by giving in succession to the radicals $\sqrt{Q}$ and $\sqrt{R}$ all the values they are, in combination, susceptible of. Thus

$$x_1 = p + \sqrt{R} + \sqrt{(q + q'\sqrt{R})},$$
$$x_2 = p - \sqrt{R} + \sqrt{(q - q'\sqrt{R})},$$
$$x_3 = p + \sqrt{R} - \sqrt{(q + q'\sqrt{R})},$$
$$x_4 = p - \sqrt{R} - \sqrt{(q - q'\sqrt{R})}.$$

By aubstituting these in the four symmetrical functions $\Sigma.x$, $\Sigma.x^2$, $\Sigma.x^3$ and $\Sigma.x^4$, equations will be found determining p, q, q' R. Here again, in computing $\Sigma.x^n$ the radicals all disappear; for whenever a radical is present with one sign in any root, there is always another root in which it is present with the opposite sign; thus these expressions in pairs cancel each other. Then' in deriving $\Sigma.x^n$, it is only necessary to preserve the terms which are free from radicals. In this way we get

$$\begin{aligned}
\Sigma.x &= 4p && = -a\,, \\
\Sigma.x^2 &= 4[p^2 + q + \mathrm{R}] && = +a^2 - 2b\,, \\
\Sigma.x^3 &= 4[p^3 + 3p(q + \mathrm{R}) + 3q'\mathrm{R}] && = -a^3 + 3ab - 3c\,, \\
\Sigma.x^4 &= 4[(p^2 + q + \mathrm{R})^2 + (4p^2 + 12pq' + q'^2)\mathrm{R} + 4q(p^2 + \mathrm{R})] \\
&= a^4 - 4a^2b + 4ac + 2b^2 - 4d\,.
\end{aligned}$$

From which we derive

$$p = -\frac{a}{4}\,, \quad q + \mathrm{R} = \frac{3a^2 - 8b}{16}\,, \quad q'\mathrm{R} = -\frac{a^3 - 4ab + 8c}{32}\,.$$

$$\mathrm{R}^3 - \frac{3a^2 - 8b}{16}R^2 + \frac{3a^4 - 16a^2b + 16ac + 16b^2 - 64d}{256}\mathrm{R} - \left(\frac{3a^3 - 4ab + 8c}{64}\right)^2 = 0\,.$$

The last is a cubic equation in R, which, by the foregoing, is solvable by radicals; hence the general equation of the fourth degree is so solvable.

In forming the value of x we may attribute to R either of the three roots of this equation. When $a = 0$, the case usually treated, the equations are simpler, viz.,

$$p = 0\,, \qquad q + \mathrm{R} = -\tfrac{1}{2}b\,, \qquad q'\mathrm{R} = -\tfrac{1}{4}c\,,$$

$$\mathrm{R}^3 + \frac{b}{2}R^2 + \frac{b^2 - 4d}{16}\mathrm{R} - \frac{c^2}{64} = 0\,.$$

If we should attempt to treat the general equation of the 5th degree in the preceding manner, we would be led to equations of higher degrees than the 5th, which must be regarded as a strong probable argument for the non-existence of an algebraic expression equivalent to the root of the general equation of this degree.

ADDITIONAL FORMULAE IN FINITE DIFFERENCES.

BY G. W. HILL.

Tne formulae I have given, (p. 144 of this Journal), give the values of the integrals $\int y dx$ and $\iint dx^2$ for the series of values of x, . . $a - h$, a, $a + h$, . .

It is generally preferable to compute them for the values, ... $a - \frac{1}{2}h$, $a + \frac{1}{2}h$, $a + \frac{3}{2}h$, ... Formulæ for this purpose can be obtained by the simple consideration, that in the scheme on p. 141, it is allowable to treat the odd orders of differences as if they were even, and the even as if they were odd.

In this way all the quantities obtained will correspond to the middle of the intervals of the former supposition. Thus, calling D^{-1} and D^{-2} in this case $D_{\frac{1}{2}}^{-1}$ and $D_{\frac{1}{2}}^{-2}$, it is evident we must have

$$D_{\frac{1}{2}}^{-1} = h\left(\int \frac{d\Delta}{\sqrt{(1+\frac{1}{4}\Delta^2)}}\right)^{-1},$$

$$D_{\frac{1}{2}}^{-2} = \frac{h^2}{\sqrt{(1+\frac{1}{4}\Delta^2)}}\left(\int \frac{d\Delta}{\sqrt{(1+\frac{1}{4}\Delta^2)}}\right)^{-2},$$

or, expanded in powers of Δ,

$$D_{\frac{1}{2}}^{-1} = h\left(\Delta^{-1} + \frac{1}{24}\Delta - \frac{17}{5760}\Delta^3 + \frac{367}{967680}\Delta^5 - \frac{27859}{464486400}\Delta^7 + \dots\right),$$

$$D_{\frac{1}{2}}^{-2} = h^2\left(\Delta^{-2} - \frac{1}{24}\Delta^0 + \frac{17}{1920}\Delta^2 - \frac{367}{193536}\Delta^4 + \frac{27859}{66355200}\Delta^6 - \dots\right).$$

The differences of the first formulæ, although they are of odd orders, are to be taken as equivalent to the simple numbers standing in the scheme on p. 141; while the differences of the second, although of even orders, are all the averages of two adjacent numbers of the same scheme

It is plain we have

$$D_{\frac{1}{2}}^{-2} = -h\frac{d\Delta_{\frac{1}{2}}^{-1}}{d\Delta}.$$

ON THE DISTRIBUTION OF PRIMES.

BY PROF. W. W. JOHNSON, ST. JOHN'S COLLEGE, ANNAPOLIS, MARYLAND.

I received a short time sinee a letter from Mr. J. W. L. Glaisher of Trinity College, Cambridge, Eng., who is engaged in reporting upon the subject of Mathematical Tables to the British Association, a short paper in which he gvies a compareson of values of the integral

$$\int_x^{x'} \frac{dx}{\log x}$$

with the actual number of primes counted between x and x'. The tables

employed are Tables of Least Divisors; Burckhardt's 1st three millions published 1814–1817, and Dase's 7th, 8th and 9th million, 1862–1865. The gap of 3 millions and the 10th million are said to exist in manuscript but have not yet been published. The primes are of course indicated in these tables by the absence of a divisor, but no separate list of primes among high numbers has been published.

Mr. Glaisher states that he has had them counted for the whole published 6 millions, and the paper gives the comparison with the formula for every interval of 50,000 in the second and also in the ninth million; the count in these cases having been executed in duplicate. There is but one actual discrepancy between these results and those given by Mr. Hill, as far as they admit of comparison. According to Mr. Glaisher the number of primes in the 3rd half-million is 35,649, error of formula —12; by Mr. Hill's table it is 35,611, error of formula +24.6. Mr. Glaisher's table however gives a much better showing for the formula than Mr. Hill's; for according to the latter there is a cumulative positive error throughout the first three millions, while according to the former the errors in the second million, (though indeed for small interval as proportionately large and and irregular as Mr.Hill points out,) show a marked tendency to balance one another.

Thus to select an extreme case, errors of — 59 and + 37 occur in adjacent 50 thousands; while however the whole error for the 2nd million is only + 9 and that for the 9th million is — 84. Mr. Glaisher states that the numbers of primes counted, for the 1st and 3rd millions, differ widely from those published by Hargrave in 1854, (who makes the descrepancies mueh greater than that given above for the 2nd million,) and he purposes soon to publish the result, for the other millions, together with a comparison with Legendre's formula.

The general correspondenc of the two formulæ may be seen thus: the differential of

$$\frac{x}{A\ log\, x - B} \quad \text{is} \quad \frac{A\ log\, x - B - A}{(A\ log\, x - B)^2}\, dx.$$

When $log\, x$ is large, and A and B near to unity, we have approximately

$$\frac{log\, x - 2}{(log\, x)^2 - 2 log\, x}\, dx = \frac{dx}{log\, x}$$

the differential of the integral formula. It is therefore to be expected that the formula

$$\frac{x}{A\ log\, x - B}$$

with values of A and B empirically determined so as to give the best possible

results within the limits of comparison selected, should within those limits give better results than

$$\int_0 \frac{dx}{logx},$$

notwithstanding the fact that the later is the true formula when the value of x is unlimited.

Prof. Hendrickson U S. Naval Academy proposes the following method of obtaining the equation of the tangent in terms of its direction ratio, the equation of the curve being in the form $y = f(x)$ . . . . (1)

Let $$m = \frac{dy}{dx} = f'(x) \quad . \; . \; . \; . \; . \; (2)$$

Then x and y being implicitly functions of m, let

$$y - mx = \psi(m) \quad . \; . \; . \; . \; . \; (3)$$

The form of the function $\psi(m)$ is reqnired. Differentiate (3), and we have $dy - mdx - xdm = \psi'(m)dm$.

Since $dy = mdx$ and dm is not zero, [(1) representing a curve]

$$-x = \psi'(m).$$

Considering x as a function of m, and integrating

$$\psi(m) = C - \int xpm.$$

To find $\psi(m)$ from (4) it is necessary to express x in terms of m from (2), perform an integration and determine C. In many cases it is easy to determine $\psi(m)$ directly from (3), which requires us to express x in terms of m as before and also to express y in terms of m by elimination from (1) and (2). When on the other hand this elimination is inconvenient, $\psi(m)$ may often be determined by (4).

The equation of the tangent is then $\quad y = mx + \psi(m)$.

EXAPLES.

1. Given $y = log\, x$, then $m = \frac{1}{x}$, whence $x = \frac{1}{m}$ and $y = -logm$.

Henc from (3) $-log\, m - 1 = \psi(m)$; and the tangeat is $y = mx - (1 + logm)$.

2. Given $y = (x^2 + 1)(x + 1)$, then $m = 3x^2 + 2x + 1$, and

$$x = -\tfrac{1}{3} \pm \sqrt{(\tfrac{1}{3}m - \tfrac{2}{9})}. \; . \; . \; (a)$$

Hence from (4) $\psi(m) = C + \frac{1}{3}m \mp 2\sqrt{(\frac{1}{3}m - \frac{2}{9})^3}$; therefore

$$y = mx + C + \tfrac{1}{3}m \mp 2\sqrt{(\tfrac{1}{3}m - \tfrac{2}{9})^3}.$$

To determine C, let $x = -1$ (say), then $y = 0$, $m = 2$; substituting [using the lower sign because, if $m = 2$ in (a), the upper sign gives $x = \frac{1}{3}$,]

$0 = -2 + C + \frac{2}{3} + \frac{16}{27}$ or $C = \frac{20}{27}$; therefore the tangentsared

$$y = mx + \tfrac{20}{27} + \tfrac{1}{3}m \mp 2\sqrt{(\tfrac{1}{3}m - \tfrac{2}{9})^3}.$$

[Prof. Johnson says that his intention was, in proposing problem 33, to require an integral equation between x and y referred to rectangular axes. The special interest, he remarks, in the problem consists in the avoidance of radicals which have not properly the double sign; and he requests us to propose the problem of finding the rectangular coordinates of the double point not on the axis of x, referring to Mr. Stille's figure in No. 9. Ed.]

FOLIATE CURVES.

BY PROF. E. W, HYDE, ITHACA, N. Y.

Prop. The foliate curves represented by the equatiou $\rho = a \cos n\theta$ (or $\rho = a \sin n\theta$) are hypotrochoids if n* be an integer, and both hypotrochoids and epitroehoids if n be fractional.

1. The equations of the hypotrochoid are

(1) $$x = (r_1 - r_2)\cos\psi + mr_2 \cos\left(\frac{r_1 - r_2}{r_2}.\psi\right)$$

(2) $$y = (r_1 - r_2)\sin\psi - mr_2 \sin\left(\frac{r_1 - r_2}{r_2}.\psi\right),$$

in which r_1 = radius of fixed circle,

r_2 = " " rolling " ,

mr_2 = distance of generating point from center of rolling circle, and ψ = angle between the axis of x and the radius of the rolling circle containing the generating point.

Let $$r_1 = pr_2 = \frac{pa}{2(p-1)}, \quad \text{and } m = p - 1.$$

Substituting in (1) and (2) we have

(3) $$x = \tfrac{1}{2}a[\cos\psi + \cos(p-1).\psi] = a\cos(\tfrac{1}{2}p.\psi)\cos[\tfrac{1}{2}(2-p).\psi],$$

(4) $$y = \tfrac{1}{2}a[\sin\psi - \sin(p-1).\psi] = a\cos(\tfrac{1}{2}p.\psi)\sin[\tfrac{1}{2}(2-p).\psi].$$

Divide (4) by (3) and we get

(5) $$\frac{y}{x} = \frac{\sin[\tfrac{1}{2}(2-p).\psi]}{\cos[\tfrac{1}{2}(2-p).\psi]} = \tan[\tfrac{1}{2}(2-p).\psi] = \tan\theta,$$ where θ equals the angle between ρ and the axis of x.

$$\therefore \tfrac{1}{2}(2-p).\psi = \theta, \text{ whence } \psi = \frac{2\theta}{2-p} \text{ and } \tfrac{1}{2}p\psi = \frac{p\theta}{2-p}.$$

Substituting these values of ψ in equation (3)

$$x = \rho\cos\theta = a\cos\frac{p\theta}{2-p}.\cos\theta,$$

* Since n, m, and p are merely numerical muitipliers they are intrinsically positive.

(6) $$\therefore \rho = a \cos \frac{p\theta}{2-q},$$

(7) $$\therefore \rho = a \cos n\theta \text{ where } n = \frac{p}{2-p}.$$

We have $p = \dfrac{2n}{n+1}$ and $a = \dfrac{2r_1(p-1)}{p} = \dfrac{r_1(n-1)}{n}$,

from which the relations of r_1, r_2 and a can be found for any value of n.

E. g. If $n = 1$, $p = 1$, $\rho = a \cos \theta$, and $a = 0$,
" $n = 2$, $p = \frac{4}{3}$, $\rho = a \cos 2\theta$, " $a = \frac{1}{2}r_1$,
" $n = 3$, $p = \frac{3}{2}$, $\rho = a \cos 3\theta$, " $a = \frac{2}{3}r_1$; &c.

Since $\cos a = \cos(-a)$ we have also

(8) $$\rho = a \cos \frac{p\theta}{p-2}.$$

Whence $p = \dfrac{2n}{n-1}$ and $a = \dfrac{r_1(n+1)}{n}$.

$\therefore$ if $n = 1$, $p = \infty$, $\rho = a \cos \theta$, and $a = 2r_1$,
" $n = 2$, $p = 4$, $\rho = a \cos 2\theta$, " $a = \frac{3}{2}r_1$,
" $n = 3$, $p = 3$, $\rho = a \cos 3\theta$, " $a = \frac{4}{3}r_1$, &c.

Also from (6) and (7)
if $n = \frac{1}{2}$, $p = \frac{2}{3}$, $\rho = a \cos \frac{1}{2}\theta$, and $a = -r_1$,
" $n = \frac{1}{3}$, $p = \frac{1}{2}$, $\rho = a \cos \frac{1}{3}\theta$, " $a = -2r_1$, &c.

Thus the proposition is proved for hypotrochoids.

2. The equations of the epitrochoid are

(9) $$x = (r_1 + r_2)\cos\phi - mr_2 \cos\left(\frac{r_1 + r_2}{r_2}.\phi\right),$$

(10) $$y = (r_1 + r_2)\sin\phi - mr_2 \sin\left(\frac{r_1 + r_2}{r_2}\phi.\right),$$

and we have $r_1 = pr_2 = \dfrac{pa}{2(p+1)}$, and $m = p + 1$.

Substituting in (9) and (10)

(11) $x = \frac{1}{2}a[\cos\phi - \cos(p+1)\phi] = -a \sin\frac{1}{2}(p+2)\phi.\sin(-\frac{1}{2}p\phi)$,

(12) $y = \frac{1}{2}a[\sin\phi - \sin(p+1)\phi] = a\cos\frac{1}{2}(p+2)\phi.\sin(-\frac{1}{2}p\phi)$.

(13) $\therefore \dfrac{x}{y} = -\tan\frac{1}{2}(p+2)\phi = \cot[\frac{1}{2}\pi + \frac{1}{2}(p+2)\phi] = \cot\phi$,

$\therefore \pi + (p+2)\phi = 2\theta$ whence

$\phi = \dfrac{2\theta - \pi}{p+2} - \frac{1}{2}p\phi = \dfrac{d(\pi - 2\theta)}{2(p+2)}$ and $\frac{1}{2}(p+2)\phi = \frac{1}{2}(2\theta - \pi) = \theta - \frac{1}{2}\pi$.

Substituaing in (11) $x = \rho\cos\theta$

$$= -a\sin(\theta - \tfrac{1}{2}\pi)\sin\frac{p(\pi - 2\theta)}{2(p+2)} = a\cos\theta\sin(\tfrac{1}{2}\pi - \theta)\frac{p}{p+2}.$$

(14) $\therefore\ \rho = a \sin\left((\tfrac{1}{2}\pi - \theta)\frac{p}{p+2}\right)$. Let $\tfrac{1}{2}\pi - \theta = \theta'$

(15) $$\rho = a \sin\frac{p}{p+2}.\theta' .$$

In this θ' being the complement of θ is to be measured from the axis of y.

Since p is essentially positive, the coefficient $p \div (p + 2)$ must be less be less than unity, and hence n must be a proper fraction.

We have $p = \dfrac{2n}{1 - n}$ and $a = \dfrac{2r_1(p+1)}{p} = \dfrac{r_1(n+1)}{n}$

$\therefore$ if $n = 1$, $p = \infty$, $\rho = a \sin \theta' = a \cos \theta$, and $a = 2r_1$,
" $n = \tfrac{1}{2}$, $p = 2$, $\rho = a \sin \tfrac{1}{2}\theta'$, " $a = 3r_1$,
" $n = \tfrac{1}{3}$, $p = 1$, $\rho = a \sin \tfrac{1}{3}\theta'$, " $a = 4r_1$,
&c., thus our demonstration is complete.

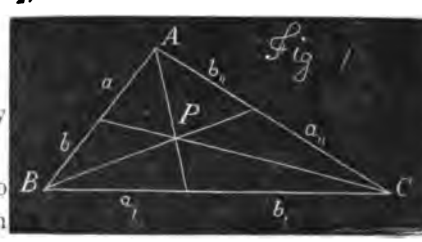
DETERMINATION OF ROOT OF N^th^ DEGREE.

BY DR. H. EGGERS, MILWAUKE, WISCONSIN.

1. I propose to give here an elementary method of extracting the root of any degree of a given number by an elementary geometrical process. This method is based on the following well known theorem of Ceva:

"A triangle ABC and an arbitrary point P in its plane are given. If we draw from the three vertices A, B, C, of the triangle three transversal lines through the point P, then on each side of the triangle two segments are formed: a and b; a_1 aud b_1; a_2 and b_2; which fulfil the relation:

$$\frac{a}{b} \cdot \frac{a_1}{b_1} \cdot \frac{a_2}{b_2} = -1"$$



This theorem solves immediately the problem:

" To constauct two lines, the ratio of which is the product of two given ratios"; which solution requires no explanation.

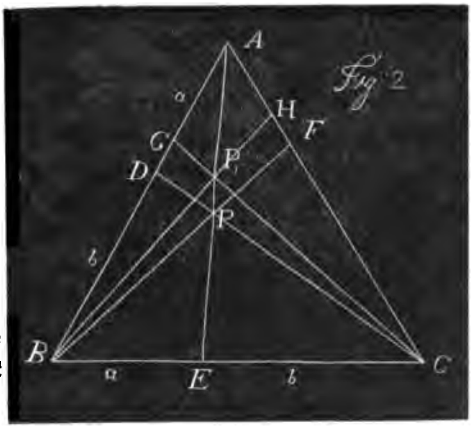
If we make the sides of the triangle equal, we can *square a given ratio*. If $a \div b$ is the given ratio, we construct an isosceles triangle with two sides equal to $a + b$. If now we draw two lines from the vertices A and C to the dividing points D and E, where AD = BE = a, and DB=EC=b, of the opposite sides, we find the point of intersection P, and the line connecting B with P marks on AC the point F, which solves the problem.

For according to Ceva's theorem we have,

$$\frac{a}{b}\cdot\frac{a}{b}=\left(\frac{a}{b}\right)^2=\frac{b_2}{a_2}.$$

In order to form the third power of the ratio $a \div b$, we have to construct an equilateral triangle with the side $(a+b)$. After having constructed according to above the square $(a \div b)^2 = \mathrm{AF} \div \mathrm{CF}$, we make the segment $\mathrm{AG} = \mathrm{AF}$ and draw the line GC, which meets AE in P_1. The line BP_1 marks on HC the point H, which point divides AC in such a manner, that

$$\frac{\mathrm{AH}}{\mathrm{CH}}=\frac{\mathrm{AG}}{BG}\cdot\frac{\mathrm{BE}}{\mathrm{CE}}=\left(\frac{a}{b}\right)^2\cdot\frac{a}{b}=\left(\frac{a}{b}\right)^3.$$



Proceeding in the same manner with AH as with AF before, we arrive at the constructiou of $(a \div b)^4$ After this we can construct the 5th powor of $a \div b$ and so on. To form the *nth* power of $a \div b$, this same process has to be repeated $n-2$ times.

2. Let us now proceed to the inverse operation, *to find the root of a given degree of a given ratio* $m \div n$.

First let us extract the square root. We construct on the side $\mathrm{AC} = m + n = \mathrm{AF} + \mathrm{FC}$ an equilatəral (or ouly an isoscele) triangle ABC, fig. 3, divide AB at an arbitrary point O, make $\mathrm{BR} = \mathrm{AO}$, draw the transversals AR and CO and through the point of intersectien P, of the two latter we draw the line BP. This line BP meets AC in some point S. If S would coincide with F then the ratio

$$\frac{\mathrm{AO}}{\mathrm{BO}}=\frac{x}{y}$$

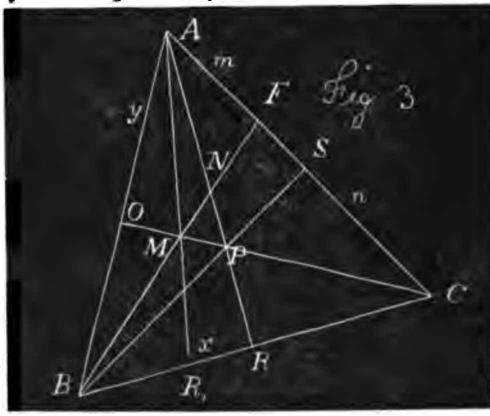
would be the exact square root of $m \div n$. But this will in general not be the case. To gain an idea of the error committed, we draw BF. This line meets AR and CO respectively iu M and N. The triangle PMN we will call the triangle of error. It is evident that the size of this triangle depends on the error of the assumed root. The area of the triangle PMN will vanish togother with the error and will increase and diminish with the error of the root. In order to find the correction of the root we draw AM which meets BC in R_1. The distance RR_1 may be called the ***segment of***

error, or briefly, the *error*, and may be designated by e, tht is, $RR_1 = e$.

In order to find a more approximate value of the root *we halve* RR_1 in X; then is

$$\frac{BX}{CX}$$

a more approximate value of the square root than the initial value $BR \div CR$.



We now try this new value by making $AY = BX$ and gain another point S_1 which will be nearer to the point F, than the point S was. Supposing the point S_1 to have a measurable distance from F, we determine as above by a new point X_1 the new error e_1, which furnishes a still more approximate value $BX_1 \div CX_1$; this process has to be repeated till the point S_k coincides in proxi with F. The last found ratio $X_k \div Y_k$ is the square root of $AF \div CF$ as near as construction can furnish it.

That this process is always convergent toward the required root needs to be demonstrated, but it would on this occasion require too much space and therefore may be left a problem for the reader. It may be stated here, that if the first trial value is any approximate value of the root, the next corrected value is so near the exact root, that the theorctical error of thea lgorithm is smaller than the error in drawing, so that a continued operation wonld be entirely useless.

3. From the above method we can immediately deduce the algorithm of finding the cube root of a given ratio. Let $m \div n$ be again the given ratio, Then we construct an equilateral triangle on the side $m + n = AC = AF + FC$. Now take an arbitrary approximate value $x \div y$ and construct the third power

$$\left(\frac{x}{}\right)^3 = \frac{AS}{CS}.$$

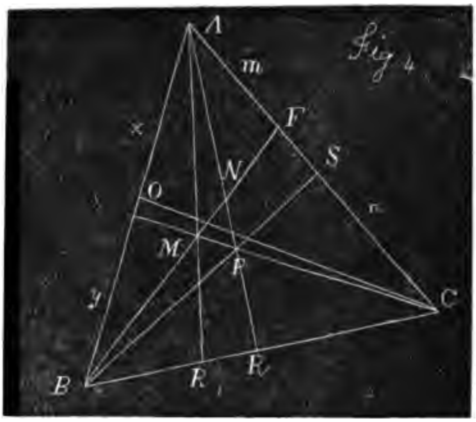
The triangle of error is PMN. Now join AM and mark the point of intersection R_1 on BC. Then is again $RR_1 = e$ the segment of error.
From this, cut off $R_1X = \frac{1}{3}R_1R$, then is

$$\frac{BX_1}{CX_1} = \frac{x_2}{y_2}$$

the correct root. If we now try this new value $x_1 \div y_1$ and should find that the new point S_1 on AC, corresponding to the point S of the first

operation be not yet sufficiently near to F we have to determine a new segment of error XR_2, and by trisecting XR_2 would find a new correcting point X_1, so that the third approximate value of the cube root would be

$$\frac{BX_1}{CX_1} = \frac{x_2}{y_2}.$$



Continued repetition of this construction furnshes a value for the cube root as approximate as required. It may be observed again, that one single construction will be sufficient; for then the error in drawing will exceed the theoretical error of the algorithm, so that in practice this construction equals or surpasses any rigorous construction by means of higher curves, even if these latter could be easily drawn.

Remark. This construction of the cube root can be applied to the renowned Delian problem: To find the side of a cube which shall be the double of a given cube. For if a be the side of a given cube, then

$$x = a\sqrt[3]{2}$$

is the side of the required cube, and we have only to extract graphically the cube root of 2, etc.

Graphical extraction of the n^{th} root of a given ratio $\frac{m}{n}$.

4. To proceed at once to the general problem to extract the root of the n^{th} degree of a given ratio $m \div n$, we construct again an equilateral triangle with the side $m + n$, or with the sum of any convenient proportionals of m and n. Then take an approximate value $x \div y$ and form of this the n^{th} power, determine in like manner as above the segment of error, e; take the n^{th} part of e, add this part to the numerator x and subtract it from the denominator y, or the reverse, as the case may be, so that the next trial value, or the corrected root, is

$$\frac{x_1}{y_1} = \frac{x + (e \div n)}{y - (e \div n)};$$

the next corrected value would be

$$\frac{x_2}{y_2} = \frac{x_1 + (e_1 \div n)}{y_1 - (e_1 \div n)} \text{ \&c.}$$

The acceleration of this algorithm toward the required root is of quadratic order.

5. This geometrical algorithm may easily be translated into an algebraical algorithm and thus furnish an elementary arithmetical method of extracting roots by a method of very rapid convergence. The general algorithm thus obtained is as follows:

$$\frac{x_1}{y_1}=\frac{(n-1)x^n+y^n z+nzxy^{n-1}}{nx^{n-1}y+x^n+(n-1)y^n z}, \tag{1}$$

where n denotes degree of root, z the given number, the root of which is to be found, $x \div y$ an arbitrary initial value and $x_1 \div y_1$ the succeeding corrected value.

By dividing the right side of (1) by y^n and substituting a for $x \div y$, and a_1 for $x_1 \div y_1$ we obtain another form of the same algorithm:

$$a_1=\frac{(n-1)a^n+z+naz}{na^{n-1}+a^n+(n-1)z} \tag{2.}$$

In regard to this algorithm the following observation may be allowed. There are the two following algorithms of a more simple form, i. e.

$$a_1=\frac{(n-1)a^n+z}{na^{n-1}} \quad (3,) \text{ and } \quad a_1=\frac{naz}{a^n+(n-1)z} \quad (4.)$$

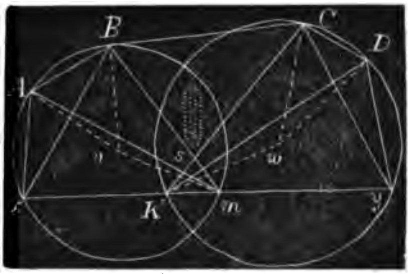
If a be again an arbitrary approximate value for the n^{th} root of z, then the corrected value a_1 in (3) always furnishes a value greater than the n^{th} root of z; and the value in (4) always is smaller than the n^{th} root of z. The acceleration of both methods is of the same (quadratic) order. Now by addng both numerators and dividing by the sum of both denominators we obtain a mean value between the two in (3) and (4); that is, another approximate value for the n^{th} root of z of at least the same degree of approximation. But this result exactly coincides with the algorithm under (2), derived from geometrical constructions.

SOLUTION OF A PROBLEM IN SURVEYING.

BY T. J. LOWRY, U. S. C. S. SAN FRANCISCO, CAL.

Problem:—Four points in the same plane being given in position to determine the position of any other two intervisible points (or places of observation) in reference to these points, having from each place of observation the angles included between the other place of observation and each of two of the known points, and with the known points so situated that the two which are visible from the first point of observation are not visible from the second, and *vice versa*.

Let A, B, C and D be the four given points, and AxB, Bxy Cyx and CyD be the observed angles.



In the $\triangle ABm$ are given AB, and the $\angle ABm$($=$ supplement of the sum of the observed $\angle$s AxB and Bxy), and the $\angle BmA$($=$ observed $\angle AxB$), we can hence find Bm. And in $\triangle DCK$ are known CD, and the $\angle DCK$ ($=$supplement of the sum of the observed $\angle$s DyC and Cyx), and the $\angle CKD$ ($=$observed $\angle CyD$) to find CK.

Now the $\angle SBC = \angle ABC - < ABm$, and $< SCB = < BCD - < DCK$ $\therefore$ in $\triangle BSC$ we have BC and all the angles to find BS and SC. But $mS = Bm - BS$, and $KS = KC - SC$ and the $< KSm = < BSC$ $\therefore$ in $\triangle$ SKm are known two sides SK and mS and the included angle, to find angles SmK and SKm. Now in $\triangle$ Bmx are given Bm, and the $<$ s Bmx and Bxm to get Bx. And then in $\triangle$ ABx we have Bx, and AB and the $< AxB$ to determine Ax. Also in $\triangle$ KCy are given KC, and $<$ s CKy and CyK to find Cy. Then in $\triangle$ CyD are known Cy, CD and the $<$ CyD to get Dy.

Since the $<$ s wDC and wCD are each equal to the compliment of the observed $< DyC$ the rule for laying down the circle of positon through C and D is obviously to lay off from CD at the points C and D the compliment of the observed $<$ CyD and the point of intersection of the produced sides of these $<$ s will be the center of a circle of position. Now about this point with radius Cw (or Dw) sweep the circle of position; and in like manner lay down the other circle of position through A and B. Then lay off from Cw at the point w twice the observed $<$ Cyx and produce the side wK till it intersects the first circle of position in some point as K (which will be a point in the line of sight through x and y): also from Bv at the point v lay off twice the observed angle Bxy and the point m where the side vm produced intersects the second circle of position will be another point in the line joining x and y. Now through K and m draw a right line and produce it each way until it intersects the circle of position and the points of intersection x and y will be the required places of observation.

But with the aid of the "three arm Protractor" this problem can be plotted much more expeditiously (and without laying down circles of position) as follows:—with the $<$ s ABx, and Bxy set off respectively on the left and right limbs of the protractor, cause the fiducial edges of the left and middle arms to traverse A, and B, and draw a line along the true edge of its right

arm, then shift the center of the protractor (taking care to keep the true edge of the left and middle arms bisecting the points A and B) and draw another line along the true edge of the right arm and the point of intersection of the two lines thus drawn will be a point in the line of sight joining x and y. Now with the $<$s DyC and Oyx set off on the right and left limbs of the protractor, shift its center till the true edges of the right, middle and left arms traverse D, C and K, dot the center and we have y (one of the places of observation). And again with the $<$s AxB and Bxy on the left and right limbs, place the true edge of the right hand arm on the line Kmy and shift the center along this line till A and B are traversed by the true edges of left and middle arms then dot the center and you have x (the other place of observation).

The Hydrographer, the Topographer and the Explorer will each find this problem servicable.

ODD NUMBERS AND EVEN NUMBERS.

BY ARTEMAS MARTIN, ERIE, PA.

All numbers are either odd or even. An *even* number is a number that can be divided by 2 without a remainder; an *odd* number is one that is not divisible by 2. 1, 3, 5, 7, are *odd* numbers; 2, 4, 6, 8, are *even* numbers.

All even numbers are comprised in the formula $2n$, and all odd numbers in either of the formulæ $2n + 1$, $2n - 1$.

Proposition I. — The sum of two even numbers is even.

Proof. — Let $2m$ and $2n$ represent any two even numbers; their sum is $2m + 2n = 2(m + n)$, which is even.

Prop. II. — The sum of two odd numbers is even.

Proof. — Let $2m + 1$ and $2n + 1$ be any two odd numbers; their sum is $2m + 1 + 2n + 1 = 2m + 2n + 2 = 2(m + n + 1)$.

Prop. III. — The sum of an odd number and an even number is odd.

Proof. — Let $2m + 1$ be any odd number and $2n$ any even number; then $2m + 1 + 2n = 2(m + n) + 1$, which is odd.

Prop. IV. — The difference of two even numbers is even.

Proof. — $2m - 2n = 2(m - n)$.

Prop. V. — The difference of two odd numbers is even.

Proof. — $2m + 1 - 2n - 1 = 2(m - n)$.

Prop. VI. — The difference of an odd number and an even one is odd.

Proof. — $2m + 1 - 2n = 2(m - n) + 1$.

If the even number is greater than the odd one, we have $2m - 2n - 1 = 2(n - m) - 1$, the second form of an odd number.

Prop. VII. — The sum of any number of *even* numbers is even.

Proof. — Let $2r_1, 2r_2, 2r_3, \ldots 2r_n$ be any number of even numbers; then $2r_1 + 2r_2 + 2r_3 + \ldots + 2r_n = 2(r_1 + r_2 + r_3 + \ldots + r_n)$, which is even.

Prop. VIII. — The sum of an *even* number of odd numbers is even.

Proof. — Let $2r_1 + 1, 2r_2 + 1, 2r_3 + 1, \ldots 2r_{2n} + 1$ be any even number of odd numbers; their sum is $2r_1 + 2r_2 + 2r_3 + \ldots + 2r_{2n} + 2n = 2(r_1 + r_2 + r_3 + \ldots + r_{2n} + n)$ an even number.

Prop. IX. — The sum of an *odd* number of odd numbers is odd.

Proof. — $(2r_1 + 1) + (2r_2 + 1) + (2r_3 + 1) + \ldots + (2r_{2n+1} + 1) = 2(r_1 + r_2 + r_3 + \ldots + r_{2n+1}) + 1$, an odd number.

Prop. X. — The product of any number of even numbers is even.

Proof. — $(2r_1)(2r_2)(2r_3) \ldots (2r_n) = 2(2^{n-1}r_1r_2r_3 \ldots r_n)$.

Prop. XI. — The product of an odd number and an even number is even.

Proof. — $2r \times (2n + 1) = 2(2rn + r)$.

Prop. XII. — The product of any number of odd numbers is odd.

Proof. — $(2r_1 + 1)(2r_2 + 1)(2r_3 + 1) \ldots (2r_n + 1) = 2b + 1$, by putting $2b$ for all the terms of the product except 1.

Some years ago the following problem was proposed in a Medical Almanac: "It is required to put 20 horses in a stable containing 5 stalls, and have an odd number of horses in every stall."

Here we have an *even* number of horses and an *odd* number of stalls, and the sum of an *odd* number of odd numbers is required to be *even*, which by *Prop.* IX, is impossible.

Many persons spent hours of patient study in vainly endeavoring to solve this impossible problem.

ON THE SOLUTION OF QUADRATIC EQUATIONS.

BY O. D. OATHOUT, READ, IOWA.

All complete quadratic equations can be made to take the form

(1) $$ax^2 + bx = c,$$

wherein a, b and c are coefficients in the most general sense of the term, and the highest exponent of x is 2. Multiply (1) by $4a$ and add b^2 to each member and we have

(2) $4a^2x^2 + 4abx + b^2 = 4ac + b^2$. Extracting square root of (2) we get

(3) $$2ax + b = \surd(4ac + b^2).$$

Now equation (3) can always be obtained directly from (1) without using (2) by observing the law of its formation, viz.; that the first member of (3) is the first derived polynomial of (1). Therefore all complete quadratic equations can be solved by the following rule:

1. Reduce to the form $ax^2 + bx = c$.

2. Make the first derived polynomial $= \pm\surd(b^2 + 4ac)$, using the double sign, and the roots of this equation are those required.

Examles. — 1. $3x^2 - 27x = -42$, to find the values of x.

The first derived polynomial is $6x - 27$; $\therefore 6x - 27 = \surd(729 - 504) = \pm 15$; $\therefore x = 7$ or 2.

2. $3\sqrt[3]{x^2} - 10\sqrt[3]{x} = -3$, to find the values of x.

Put $\sqrt[3]{x} = y$; then $\sqrt[3]{x^2} = y^2$. Therefore the given equation becomes $3y^2 - 10y = -3$. First derived polynomial is $6y - 10$, $\therefore 6y - 10 = \surd(100 - 36) = \pm 8$; $\therefore y = 3$ or $\frac{1}{3}$; $\therefore x = 27$ or $\frac{1}{27}$.

3. $\frac{1}{3}x^2 + 5x = 42$, to find the value of x.

The first derived polynomial is $\frac{2}{3}x + 5$; $\therefore \frac{2}{3}x + 5 = \surd(25 + 56) = \pm 9$; $\therefore x = 6$ or -21.

It will be seen that if the first derived polynomial is made equal to 0, the resulting value of x is that required for a maximum or a minimum.

The works of Davies, Robinson, Ray and Schuyler do not contain this method of solving quadratics. I discovered it last October; but it is probable that the method has long been known, though it may never have been published.

SOLUTION OF A PROBLEM.

BY THEO. L. DE LAND, WASHINGTON, D. C.

Supposing the sky, all the way from the zenith to the horizon, to be thickly dotted with stars, and that they are equally distributed in every part, what would be the mean of all their altitudes?

Solution. — Let n be the number of stars, h the altitude of any one of them, and A its azimuth.

The area of an elementary portion of the sky is $\cos h.dhdA$ which must be taken as constant.

$$\text{Mean altitude} = \frac{h_1 + h_2 + h_3 + \ldots + h_n}{n}.$$

Multiplying both numerator and denominator by the above factor, bearing in mind that the denominator then becomes the whole sky, we have

$$\text{Mean altitude} = \frac{\int_0^{2\pi}\int_0^{\frac{\pi}{2}} h \cos h\, dh dA}{2\pi}$$

$$= \int_0^{\frac{\pi}{2}} h \cos h dh \qquad = \Big[h \sin h + \cos h \Big]_0^{\frac{\pi}{2}}$$

$$= \tfrac{1}{2}\pi - 1.$$

$\frac{1}{2}\pi$ in angular measure is 90°; and 1 is an arc equal to the radius.

$\therefore$ Mean altitude $= 90° - 57°\ 17'\ 44.8'' = 32°\ 42'\ 15.2''$.

SOLUTON OF A PROBLEM.

BY PROF. C. M. WOODWARD, ST. LOUIS, MO.

To find in terms of the coordinates of a point (x',y') on one branch of the equilateral hyperbola $xy = m$, the length of a normal dropped from (x',y') upon the other branch.

Let the equation to the normal be

$$\frac{x - x'}{c} = \frac{y - y'}{s} = l$$

or $\qquad x = cl + x' \qquad$ and $y = sl + y'$.

Multiplying we have $\quad xy = l^2sc + l(x's + y'c) + x'y'$, but $xy = m$, and $x'y' = m$ for the intersections, hence $l^2sc + l(x's + y'c) = 0$.

$$l = \frac{x's + y'c}{sc} = \frac{x'}{c} + \frac{y'}{s} = x' \sec\theta + y' \operatorname{cosec}\theta.$$

(neglecting the negative sign, as that has reference only to the direction of measurement).

Now this distance must be a minimum; so put

$$\frac{dl}{d\theta} = x' \tan\theta \sec\theta - y' \cot\theta \cos\theta = 0.$$

$$\tan^3\theta = \frac{y'}{x'} \qquad \tan\theta = \frac{\sqrt[3]{y'}}{\sqrt[3]{x'}}$$

$$\sec\theta = \frac{\sqrt{(\sqrt[3]{x'^2} + \sqrt[3]{y'^2})}}{\sqrt[3]{x'}} \qquad \operatorname{cosec}\theta = \frac{\sqrt{(\sqrt[3]{x'^2} + \sqrt[3]{y'^2})}}{\sqrt[3]{y'}},$$

$$\therefore l = x'^{\frac{2}{3}}\left(x'^{\frac{2}{3}} + y'^{\frac{2}{3}}\right)^{\frac{1}{2}} + y'^{\frac{2}{3}}\left(x'^{\frac{2}{3}} + y'^{\frac{2}{3}}\right)^{\frac{1}{2}} = \left(x'^{\frac{2}{3}} + y'^{\frac{2}{3}}\right)^{\frac{3}{2}}.$$

SOLUTION OF A PROBLEM.

BY ISAAC H. TURRELL, CUMMINSVILLE, OHIO.

Given three circles touching each other externally; if another group of three be drawn touching each other in like manner, and so situated that each circle of either group touches two of the other, to prove that the square of the common tangent of any two of the circles, which do not touch each other, is equal to twice the product of their diameters.

Solution. This property is easily proved by means of the following principle, due to Mr. Casey, and found on page 113, Salmon's Conics, Fourth Edition: "If four circles be all touched by the same fifth circle, the lengths of their common tangents are connected by the following relation: $\overline{12}\,.\,\overline{34} \pm \overline{14}\,.\,\overline{23} \pm \overline{13}\,.\,\overline{24} = 0$, where $\overline{12}$ denotes the length of a common tangent to the first and second circles, &c."

Denoting the circles of the first group and their radii by a, b, c, and of the second by x, y, z, there will be three pairs of non-touching circles, viz; a,x, b,y, and c,z., Let T, T', T'' be their common tangents, then as z is touched by a, b, x, y, which touch each other consecutively,

$$2\sqrt{(ab)}\times 2\sqrt{(xy)} + 2\sqrt{(ay)}\times 2\sqrt{(bx)} = T.T',$$

or $\quad 8\sqrt{(abxy)} = T.T'.$

Similarly $\quad 8\sqrt{(bcyz)} = T'.T'',$

and $\quad 8\sqrt{(cazx)} = T''.T,$ whence, by elimination,

$T^2 = 8ax =$ twice the product of the diameters of a and x.

NOTE.—TO THE STUDENT OF PHYSICAL ASTRONOMY: When you undartake the study of the theory of Perturbations of the planetary motions, do not take up any extended work on the subject and attempt to master all its details at one reading, but try to get the *general principles* of the subject — not the most general methods — and as soon as you have done that, commence to prepare with your own hands a treatise on the subject for your own use. Continue to *think* on the subject till you can explain to another all the details so far as you go, and do this in your treatise. By all means commence with the *Lunar Theory*, and I should say the "*Variation of Constants*".

DAVID TROWBRIDGE.

SOLUTION OF PROBLEMS IN NUMBERS 11 *AND* 12, *VOL.* 1.

Solutions of problems in Nos. 11 and 12 have been received as follows:

From R. J. Adcock, 48; A. L. Baker, 44, 45 & 50; G. L. Dake, 41, 43 & 45; Theo. L. De Land, 41, 44, 45 & 50; A. B. Evans, 48 & 50; Henry Gunder, 42, 43, 44, 45 & 46; Phil. Hoglan, 41, 42 & 45 Max. Liporwitz, 41; Artemas Martin, 41, 42, 43, 44, 45 & 48; Walter Siverly, 43 & 50.

41.—"Given $x^2 + y = 7$ . . . (1), $x + y^2 = 11$ . . . . . . (2), to find the values of x and y."

SOLUTION BY MAX. LIPORWITZ, CRESCENT CITY, CAL.

Eliminating x^2 in (1) and y^2 in (2), we have

(3) $x^4 - 14x^2 + x + 38 = 0$, and

(4) $y^4 - 22y^2 + y + 114 = 0$.

On inspection, one of the roots of (3) is found to be 2; and, in like manner, one of (4) to be 3. Freeing (3) and (4) of these roots, the equations are

(5) $x^3 + 2x^2 - 10x - 19 = 0$, and

(6) $y^3 + 3y^2 - 13y - 38 = 0$. Now, making

(7) $x = z - \frac{2}{3}$, and

(8) $y = u - 1$, and substituting these values of x and of y in (5) and (6), we obtain

(9) $z^3 - \frac{34}{3}z - \frac{317}{27} = 0$, and

10) $u^3 - 16u - 23 = 0$.

Applying the method of circular functions to the solution of (9) and of (10), we find the roots of z and of u which latter substituted in (7) and (8), furnish the three additional roots of x and of y, so that

$x = 2$,	3.131313,	$-$ 3.283185,	or	$-$ 1.848120,
and $y = 3$,	$-$2.805118,	$-$ 3.779095,	or	3.584284.

SOLUTION BY L. REGAN, BOONSBORO, IOWA.

From (1) we get $x = \sqrt{(7 - y)}$; substitute this value for x in (2) and we have $y^2 + \sqrt{(7 - y)} = 11$, or $y^2 - 11 = -\sqrt{(7 - y)}$. Squaring both sides of this equation we get $y^4 - 22\,y^2 + y + 114 = 0$, or, $y^4 - 13y^2 + 39y = 9y^2 + 38y - 114$; by adding $3y^3$ to each side and factoring we have $y(y^3 + 3y^2 - 13y - 38) = 3(y^3 + 3y^2 - 13y - 38)$; dividing by the common factor we get $y = 3$, and from (2) we get $x = 2$.

42.—"Find two integral numbers the difference of whose squares is a cube, and the difference of their cubes a square."

SOLUTION BY ARTEMAS MARTIN, ERIE, PA.

Let $ax^3 + b$ and $ax^3 - b$ denote the numbers.

The difference of their squares is $4abx^3$, which must be a cube, $= 8a^3x^3$, then $b = 2a^2$.

The difference of their cubes is $6a^2bx^6 + 2b^3$, which must be a square, or

$$3x^6 + 4a^3 = \square = (2x^3 - 2a)^2. \quad \text{Whence } x^3 = 8a.$$

Take $a = 1$, then $x = 2$, $b = 2$, and the numbers are 10 and 6.

43.—"Let AB and AC be two lines intersecting each other at right angles in A, and D, any point given in position. Required the position of a line EF through D intersecting the lines AB and AC in E and F, so that $(DE)^2 + (DF)^2 = m^2$."

SOLUTION BY E. B. SEITZ, GREENVILLE, O.

Draw DH perpendicular to AB, and DK to AC. Put $DH = a$, $DK = b$, $\angle DEA = \theta$. Then $DE = a \operatorname{cosec} \theta$, and $DF = b \sec \theta$;

$$\therefore a^2 \operatorname{cosec}^2\theta + b^2 \sec^2\theta = m^2, \text{ or } a^2\left(\frac{1 + \tan^2\theta}{\tan^2\theta}\right) + b^2(1 + \tan^2\theta) = m^2,$$

or $$b^2 \tan^4\theta - (m^2 - a^2 - b^2)\tan^2\theta = - a^2, \text{ whence}$$

$$\tan^2\theta = \frac{m^2 - a^2 - b^2 \pm \sqrt{[(m^2 - a^2 - b^2) - 4a^2b^2]}}{2b^2}, \text{ or}$$

$$\tan\theta = \pm\frac{1}{2b}\left[\sqrt{(m + a - b)(m - a + b)} \pm \sqrt{(m + a + b)(m - a - b)}\right]$$

which determines the angle that the line EF makes with AB. There are, therefore, in general, four positions of the line.

44.—"If a circle be divided into three equal parts by two parallel chords, find the perpendicular distance between the chords in terms of the radius."

SOLUTION BY PROF. J. SCHEFFER, KENYON COLLEGE, GAMBIER, O.

Let C be the center of the circle and AB a chord which cuts off its third part. Denoting the angle ACB by ϕ, we obtain for the area of the segment, the expression $r^2\pi\phi \div 360 - \frac{1}{2}r^2 \sin\phi$. Hence we have the equation

$$\frac{r^2\pi}{390}\phi - \frac{r^2 \sin\phi}{2} = \frac{\pi r^2}{3}, \text{ or } \phi - \frac{180}{\pi}\sin\phi - 120 = 0,$$

an equation which pertains to that class of equations called transcendental equations. Such equations can only be solved by a process of approximation. Thus we find $\phi = 149° \; 16' \; 30''$. Hence the required distance equals $2r \cos \frac{1}{2}\phi = .52985r$.

45.—"Required the area and sides of an obtuse angled triangle whose angles are to one another as 2, 3 and 7 and whose longest side equals 1. No logarithms to be used in the solution."

SOLUTION BY HENRY GUNDER, GREENVILLE, O.

We readily find the angles to be 30°, 45°, and 105°.

Demitting a perpendicular from the 105° $\angle$, and calling it x, then will the other sides and area be $2x$, $x\sqrt{2}$ and $\frac{1}{2}x$. From the two right triangles thus formed we get $x(\sqrt{3}+1) = 1$, $\therefore x = \frac{1}{2}(\sqrt{3}-1)$. Hence the sides are $\sqrt{3}-1$, and $\frac{1}{2}(\sqrt{6}-\sqrt{2})$ and the area is $\frac{1}{4}(\sqrt{3}-1)$.

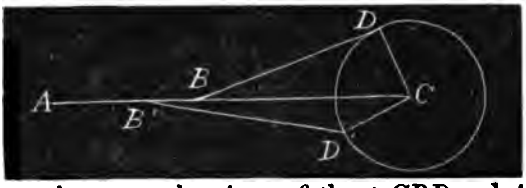
46.—"The area of the piston of a steam engine is 1200 square inches, the length of stroke $8\frac{1}{2}$ ft., the pressure of steam upon the piston 32 lbs. per sq. inch and the number of strokes per min. 18. Required the number of cubic feet of water the engine will raise from a mine 60 fath. deep, the friction being estimated at 1 lb. per sq. inch plus the pressure of the atmosphere."

SOLUTION BY HENRY GUNDER.

The moving force $= 32 - 15.7225 = 16.5775$ lbs. per sq. inch. Therefore $(1200 \times 16.2775 \times 18 \times 8\frac{1}{2}) \div (92.418 \times 360) = 133$ — cubic feet per minute.

This seems to me to be the common sense view of it, since there is nothing said about the expansion of the steam during the stroke of the piston, nor its cooling. [We inserted this question by special request, though it was obvious that the data are insufficient for a practical solution].

47.—[No solution of this question has been received.—The inertia of a body in motion may be represented by the force that would be required to arrest the motion.



Let AB represent the piston, BD, $B'D'$ the connecting rod and CD, CD' the crank arm in two different positions. Then, if f represents the force acting upon the piston, θ the $\angle\, CBD$ and ϕ the $\angle\, BCD$, the relative moving force on a point in the axis of the

connecting rod will be represnted by the formula $f\cos\theta\sin\phi$.

The inertia of a point in the axis of the connecting rod, and in the line of that axis, will therefore be a maximum when $\cos\theta\sin\phi$ is a maximum. The position of the connecting rod which corresponds to its maximum inertia in the line of its axis, depends, therefore, upon the relative length of the crank arm and the connecting rod.—ED.]

48.—A cylindrical tower, radius r, is surrounded by a walk, width a. Two persons are on the walk; what is the probability that they can see each other?"

SOLUTION BY ARTEMAS MARTIN, ERIE, PA.

Let P be the position of one of the persons. Through P draw the diameter $APOB$, and from P draw PG, PH tangent to the tower at C and D; then if the other person be on the surface $AEGCIDHFA$ they can see each other.

Let $OP = x$, then arc EA = arc AF $= (r+a)\cos^{-1}\left(\frac{r}{x}\right)$, arc CI = arc ID $= r\cos^{-1}\left(\frac{r}{x}\right)$, arc GE = arc $FH = (r+a)\cos^{-1}\left(\frac{r}{r+a}\right)$, $GC = DH = \sqrt{(2ar+a^2)}$; area ECG = area $FDH = \frac{1}{2}(r+a)^2 \times\cos^{-1}\left(\frac{r}{r+a}\right) - \frac{1}{2}r\sqrt{(2ar+a^2)}$, area $EACI$ = area $AFID = \frac{1}{2}(r+a)^2 \times\cos^{-1}\left(\frac{r}{x}\right) - \frac{1}{2}r^2\cos^{-1}\left(\frac{r}{x}\right) = \frac{1}{2}(2ar+a^2)\cos^{-1}\left(\frac{r}{x}\right)$.

$\therefore p =$

$$\frac{\int_r^{r+a}\left[(r+a)^2\cos^{-1}\left(\frac{r}{r+a}\right)+(2ar+a^2)\cos^{-1}\left(\frac{r}{x}\right)-r\sqrt{(2ar+a^2)}\right]2\pi x\,dx}{\pi(2ar+a^2)\int_r^{r+a}2\pi x\,dx}$$

$$=\frac{2(a+r)^2\cos^{-1}[r\div(a+r)]}{\pi(2ar+a^2)}-\frac{2r}{\pi\sqrt{(2ar+a^2)}}.$$

49.—"How far will a man travel in unwinding an inch rope from a frustrum of a cone whose upper diameter is 2 ft., lower 15 ft. and height 35 ft. the rope to be closely wound around the frustrum from top to bottom?"

Solution by Prof. *J. Scheffer*, Gambier, Ohio.

The area of the surface of the frustrum is $\pi(R + r)\surd[h^2 + (R + r)^2]$, R denoting the radius of the lower base, r that of the upper and h the axis, of the cone. The question arises now; what must be the length of a rectangular strip whose breadth is one inch, in order that its area may be equal to the above stated surface? Thus we find for the required distance:

$12\pi(R + r)\surd[h^2 + (R + r)^2] = 7616.6\pi = 23928.3$ ft., or nearly $4\frac{1}{2}$ miles.

This solution is not mathematically accurate. The line which the rope forms is not a curve (spiral or screw line) of double curvature, in fact it is no mathematical curve at all whose length we could find by means of the formulas of Calculus. The distance, however, which has been found in the above simple and elementary way, will not differ by a great deal from the *actual* one.

[It will be seen that, in the above solution, Prof. Scheffer has estimated approximately, the length of the *rope*, (or, rather, double the length, as, in substituting for R and r he has probably used the *diameters* of the bases instead of their radii,) but he has not attempted the more difficult part of the question, viz; to find the length of the involute curve traced on the ground in unwinding the rope.

Let r represent the distance from the apex of the cone to any point P, on its surface where the distance between the centers of two consecutive coils is x; and let θ represent the angle traversed by the radius vector r in describing that part of the curve which lies above the point P, then is $r = \int dxb\theta$ the polar equation of the curve. If x were constant this equation would represent the Spiral of Archimedes. For the given frustrum x is nearly constant and nearly equal to one inch, but not quite.

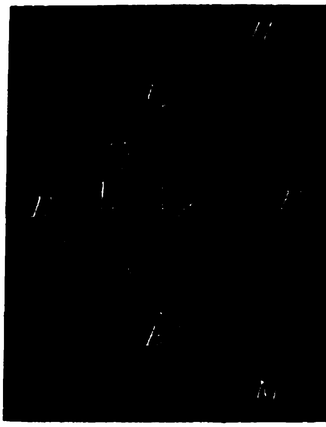
Determine x in functions of θ; then is $r = \int d.\psi(\theta)$. . . . . . (1)
From (1) determine the length of the curve between the limits $r = h$ and $r = k$. (h representing the slant height of the complete cone and k the distance, on the slant side, from the apex of the cone to any point P, to which the rope may have been unwound). Let L represent the length between the above named limits and let p represent the perpendicular height of the point P; then will $\surd(L^2 + p^2) =$ the radius of curvature of the involute curve, from which its length may be found.

As a first step in the actual solution of prob. 49 we propose the question: To find the equation between x and θ as involved in (1) above.—Ed.]

50.—"Assuming the earth's orbit to be a circle, if a comet move in a parabola around the sun and in the plane of the earth's orbit, show that the comet cannot remain within the earth's orbit longer than 78 days."

Solution by Prof. *A. B. Evans*, Lockport, N. Y.

Let MAN be the orbit of a comet whose time within the earth's orbit $BEDF$ is a maximum. Let A be the comet's position when nearest the sun S, and C any other position of the comet in its orbit.



Put $SE = r$, $SA = a$, $SC = \rho$, $\angle ASC = \theta$: then the equation of the parabola is

$$\rho = \frac{2a}{1+\cos\theta} = \frac{a}{\cos^2\frac{1}{2}\theta},$$ and the area

$$ASC \text{ is } \tfrac{1}{2}\int\rho^2 d\theta = \tfrac{1}{2}a^2\int\frac{d\theta}{\cos^4\frac{1}{2}\theta}$$

$$= \tfrac{1}{2}a^2\int(1+\tan^2\tfrac{1}{2}\theta)\sec^2\tfrac{1}{2}\theta d\theta$$

$$= a^2\tan\tfrac{1}{2}\theta + \tfrac{1}{3}a^2\tan^3\tfrac{1}{2}\theta. \quad . \quad . \quad . \quad (1)$$

Since $\cos^2\frac{1}{2}\theta = a\div\rho$, the area of ASD is found by putting $\cos\frac{1}{2}\theta = \sqrt{(a\div r)}$, or $\tan\frac{1}{2}\theta = \sqrt{[(r-a)\div a]}$, in (1). The area of ASD is therefore equal to $a^2\sqrt{[(r-a)\div a]} + \frac{1}{3}a^2\sqrt{[(r-a)^3\div a^3]}$ $= \frac{1}{3}\sqrt{a}\times(r+2a)\sqrt{(r-a)}$; and the area of $BSA + ASD = 2ASD$

$$= \tfrac{2}{3}\sqrt{a}\times(r+2a)\sqrt{(r-a)}. \quad . \quad . \quad . \quad (2)$$

Now the areas described about a common center of force by two bodies moving in different orbits being in the subduplicate ratio of the parameters of those orbits, and the parameters of the orbits in this case being $2r$ and $4a$, we have $\sqrt{(2r)} : \sqrt{(4a)} :: \pi r^2 : \pi\sqrt{r^3}\times\sqrt{(2a)}$ = the area described by the comet in one year. As the comet describes equal areas in equal times, we have, denoting the number of days in the year by T,

$\pi\sqrt{r^3}\sqrt{(2a)} : \frac{2}{3}\sqrt{a}.(r+2a)\sqrt{(r-a)} :: T : T'$ the number of days the comet is within the earth's orbit

$$\therefore\ T' = \frac{(r+2a)\sqrt{(2r-2a)}}{3\pi\sqrt{r^3}}T \quad . \quad . \quad . \quad . \quad (3)$$

The only variable in the value of T' being a, T' will be a maximum when $(r+2a)\sqrt{(2r-2a)}$ is a maximum. Differentiating this expression with respect to r and placing the differential coefficient equal to zero, we find $2\sqrt{(2r-2a)} - (r+2a)\div\sqrt{(2r-2a)} = 0$; whence $r = 2a$. This value of r substituted in (3) gives

$$T' = \frac{2}{3\pi}.T = \frac{2}{3}\left(\frac{365.2563612}{3.14159}\right) = 77.5 \text{ days } \therefore T' < 78 \text{ days.}$$

PROBLEMS.

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51. *By Dr. N. R. Oliver, Sacaton, Arizona.*—Given

$$(x^2 + y^2)y = 39, \quad (1)$$

$$x^4 + y^4 = 97, \quad (2)$$

to find x and y by quadratics.

52. *By J. M. Curtis, Ovid, Mich.*—A hare starts 10 rods north of a hound and runs at an angle of 45° with the meridian. The hound pursues in a straight line and in such a direction that he will intercept the hare without changing his course. Supposing the hound to run n times as fast as the hare, how far will he run before he catches the hare?

43. By request of *A. W. Mason.*—If a ball of 6 in. diameter is discharged from a canon at the rate of 1 mile in 7 seconds, how much greater force will be required to throw a ball of double the weight with the same velocity, taking into account the resistance of the air and the diameters of the balls?—*Silliman's Physics*, page 108.

54. By Prof. *J. Scheffer*, Gambier, O.—Let ABC represent a spherical triangle and M the centre of the sphere. Find the three distances of the centre of gravity of the spherical triangle ABC from the three planes ABM, ACM and BCM.

55. By request of *Walter Siverly.*—A very small bar of matter is moveable about one extremity which is fixed half way between two centers of force attracting inversely as the square of the distance; if l be the length of the bar, and $2a$ the distance between the centers of force, prove that there will be two positions of equilibrium for the bar, or four, according as the ratio of the absolute intensity of the more powerful force to that of the less powerful is or is not greater than $(a + 2l) \div (a - 2l)$: and distinguish between the stable and unstable positions.—*Cambridge Problems*, 1845.

56. By *G. W. Hill.*—Find the real value of $(\sqrt{-1})^{\sqrt{-1}}$.

57. By Prof. *E. W. Hyde.*—Find the nature of the curve represented by the equation

$$4b^2(x^2 + y^2)[(x-a)^2 + y^2 - b^2] - a^2y^2[(x-a)^2 + y^2] = 0,$$

and show that when a certain relation exists between a and b, the locus reduces to a right line perpendicular to the axis of x, and a 3rd degree curve which is a trisectrix.

58. By Prof. *W. W. Johnson.*—Referring to prob. 33, and Mr. Stille's figure in No. 9, find the rect. coordinates of the doub. pt. not on the axis of x.

ERRATA.

PAGE.	LINE.					
3	10	from bottom,	for	"$\psi''x$"	read	$\frac{1}{2}\psi''x$.
4		first line	"	$\frac{1}{x}\sqrt{(1-x^2)}$	read	$\frac{1}{x}\sqrt{(1-x)}$.
"	4		"	$\frac{x}{\sqrt{(1-x^2)}}$	"	$\frac{x^2}{\sqrt{(1-x^2)}}$.
"	7		"	$(x+h)e^{-x+h}$	"	$(x+h)e^{-x-h}$,
6	9		"	$x_1x_2x_3=o$	"	$x_1x_2x_3=-c$.
"	3	from bottom,	"	a^23+b	"	a^2-3b.
7	3 & 7		"	A^2	"	A^3.
8		last line,	"	$\int\int dx^2$	"	$\int\int ydx^2$.
9	9	from bottom,	"	$-h\frac{d\Delta_{\frac{1}{2}}^{-1}}{d\Delta}$	"	$-h\frac{dD_{\frac{1}{2}}^{-1}}{d\Delta}$.
9	6	from bottom,	"	*dele* "a letter".		
10	7		"	"them"	read	the primes.
"	25		"	Hargrave	"	Hargreave.
11	17		"	$C-\int xpm$	"	$C-\int xdm$. . . . (4)
13		first line	"	$2-q$	"	$2-p$.
"	3	from bottom,	"	$d(\pi-2\theta)$	"	$p(\pi-2\theta)$.
14	8		"	θ^7	"	θ'.
16	16		"	"proxi"	"	practice.
"	7	from bottom,	"	$\left(\frac{x}{\ }\right)^3$	"	$\left(\frac{x}{y}\right)^3$.

We regret that, owing to a new arrangement for printing the *Analyst* and a new compositor, and other sufficient reasons which need not be explained, the present No. appears with an unusually large number of typographical errors. We apologise to the authors for the appearance of their articles and hope to be able to bestow such personal attention to the proof reading, in the future, as will obviate the necessity of a like apology hereafter.

In the above *errata* we have, with two or three exceptions, given only the corrections of erroneous formulas that have been observed, without noticing the errors in spelling and punctuation which are very numerous: the reader, however, will correct these at a glance, and the authors will pardon our apparent inattention in this matter, we hope, for once at least.—ED.

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PERFECT CUBES.

BY PROF. W. D. HENKLE, SALEM, OHIO.

In my first article on this subject (See ANALYST Vol. I. No. 5.) I explained how the tens' figure of the root of a perfect cube can be obtained from the tens' figure of the cube. In this article I shall extend the discussion to the hundreds' figure of the root. In order not to use too much space in the ANALYST I adopt a brief mode of indicating the facts, which may be readily comprehended by a reference to the former article. The figures after the braces are the endings of the perfect cubes and the subs the endings of the roots. The figures in the upper rows before the braces represent the hundreds of the cube and those in the lower rows the corresponding hundreds of the root. The direction after each brace refers to the mode of obtaining any figure in the lower from the corresponding figure in the upper row. (I use for brevity the word *figure* instead of *number represented by the figure*). In each case of multiplication or of multiplication and addition the tens are to be cast out or the left-hand digit is to be rejected from the product or sum.

$$\left.\begin{array}{cccccccccc}1&2&3&4&5&6&7&8&9&0\\7&4&1&8&5&2&9&6&3&0\end{array}\right\}01_{01}\text{; multiply by 7 and cast out the tens.}$$

$$\left.\begin{array}{cccccccccc}1&2&3&4&5&6&7&8&9&0\\4&1&8&5&2&9&6&3&0&7\end{array}\right\}\begin{array}{l}11_{71}\\21_{41}\\81_{61}\end{array}\text{; mult. by 7, add 7, and cast out the tens.}$$

$$\left.\begin{array}{cccccccccc}1&2&3&4&5&6&7&8&9&0\\6&3&0&7&4&1&8&5&2&9\end{array}\right\}31_{11}\text{; mult. by 7, add 9, and cast out the tens.}$$

$$\left.\begin{array}{cccccccccc}1&2&3&4&5&6&7&8&9&0\\9&6&3&0&7&4&1&8&5&2\end{array}\right\}41_{81}\text{; mult. by 7, add 2, and cast out the tens.}$$

$$\left.\begin{array}{cccccccccc}1&2&3&4&5&6&7&8&9&0\\5&2&9&6&3&0&7&4&1&8\end{array}\right\}51_{51}\text{; mult. by 7, add 8, and cast out the tens.}$$

$$\left.\begin{array}{cccccccccc}1&2&3&4&5&6&7&8&9&0\\3&0&7&4&1&8&5&2&9&6\end{array}\right\}61_{21}\text{; mult. by 7, add 6, and cast out the tens.}$$

1	2	3	4	5	6	7	8	9	0	71_{91};	× by 7, add 5, and cast out the tens.
2	9	6	3	0	7	4	1	8	5		
1	2	3	4	5	6	7	8	9	0	91_{31};	× by 7, add 1, and cast out the tens.
8	5	2	9	6	3	0	7	4	1		
1	2	3	4	5	6	7	8	9	0	03_{87}, 33_{77};	× by 3, add 5, and cast out the tens.
8	1	4	7	0	3	6	9	2	5		
1	1	3	4	5	6	7	8	9	0	13_{17};	× by 3, add 3, and cast out the tens.
6	9	2	5	8	1	4	7	0	3		
1	2	3	4	5	6	7	8	9	0	23_{47};	× by 3, add 6, and cast out the tens.
9	2	5	8	1	4	7	0	3	6		
1	2	3	4	5	6	7	8	9	0	43_{07};	× by 3, add 1, and cast out the tens.
4	7	0	3	6	9	2	5	8	1		
1	2	3	4	5	6	7	8	9	0	53_{37}, 73_{97}, 83_{27};	× by 3, add 2, and cast out the tens.
5	8	1	4	7	0	3	6	9	2		
1	2	3	4	5	6	7	8	9	0	63_{67};	× by 3, add 9, and cast out the tens.
2	5	8	1	4	7	0	3	6	9		
1	2	3	4	5	6	7	8	9	0	93_{57};	× by 3, add 7, and cast out the tens.
0	3	6	9	2	5	8	1	4	7		
1	2	3	4	5	6	7	8	9	0	07_{43};	× by 3, add 5, and cast out the tens.
8	1	4	7	0	3	6	9	2	5		
1	2	3	4	5	6	7	8	9	0	17_{73}, 27_{03}, 47_{63};	× by 3, add 0, and cast out the tens.
3	6	9	2	5	8	1	4	7	0		
1	2	3	4	5	6	7	8	9	0	37_{33};	× by 3, add 3, and cast out the tens.
6	9	2	5	8	1	4	7	0	3		
1	2	3	4	5	6	7	8	9	0	57_{93};	× by 3, add 1, and cast out the tens.
4	7	0	3	6	9	2	5	8	1		
1	2	3	4	5	6	7	8	9	0	67_{23}, 97_{13};	× by 3, add 7, and cast out the tens.
0	3	6	9	2	5	8	1	4	7		
1	2	3	4	5	6	7	8	9	0	77_{53};	× by 3, add 6, and cast out the tens.
9	2	5	8	1	4	7	0	3	6		
1	2	3	4	5	6	7	8	9	0	87_{83};	× by 3, add 9, and cast out the tens.
2	5	8	1	4	7	0	3	6	9		
1	2	3	4	5	6	7	8	9	0	09_{69},	× by 7, add 5, and cast out the tens.
2	9	6	3	0	7	4	1	8	5		
1	2	3	4	5	6	7	8	9	0	19_{39}, 79_{59}, 89_{29};	× by 7, add 9, and cast out the tens.
6	3	0	7	4	1	8	5	2	9		

$$\left.\begin{array}{cccccccccc}1&2&3&4&5&6&7&8&9&0\\8&5&2&9&6&3&0&7&4&1\end{array}\right\}29_{09};\ \times\text{ by 7, add 1, and cast out the tens.}$$

$$\left.\begin{array}{cccccccccc}1&2&3&4&5&6&7&8&9&0\\7&4&1&8&5&2&9&6&3&0\end{array}\right\}39_{79};\ \times\text{ by 7, add 0, and cast out the tens.}$$

$$\left.\begin{array}{cccccccccc}1&2&3&4&5&6&7&8&9&0\\5&2&9&6&3&0&7&4&1&8\end{array}\right\}49_{49},\ \times\text{ by 7, add 8, and cast out the tens.}$$

$$\left.\begin{array}{cccccccccc}1&2&3&4&5&6&7&8&9&0\\1&8&5&2&9&6&3&0&7&4\end{array}\right\}59_{19};\ \times\text{ by 7, add 4, and cast out the tens.}$$

$$\left.\begin{array}{cccccccccc}1&2&3&4&5&6&7&8&9&0\\4&1&8&5&2&9&6&3&0&7\end{array}\right\}69_{89};\ \times\text{ by 7, add 7, and cast out the tens.}$$

$$\left.\begin{array}{cccccccccc}1&2&3&4&5&6&7&8&9&0\\3&0&7&4&1&8&5&2&9&6\end{array}\right\}99_{99};\ \times\text{ by 7, add 6, and cast out the tens.}$$

$$\left.\begin{array}{cccccc}&1&3&5&7&9\\&3&9&5&1&7\\\text{or}&8&4&0&6&2\end{array}\right\}\begin{array}{l}12_{08}\\52_{78}\\92_{48}\end{array};\ \times\text{ by 3, add 0 or 5, and cast out the tens.}$$

$$\left.\begin{array}{cccccc}&1&3&5&7&9\\&5&1&7&3&9\\\text{or}&0&6&2&8&4\end{array}\right\}\begin{array}{l}12_{58}\\92_{98}\end{array};\ \times\text{ by 3, add 2 or 7, and cast out the tens.}$$

$$\left.\begin{array}{cccccc}&2&4&6&8&0\\&7&3&9&5&1\\\text{or}&2&8&4&0&6\end{array}\right\}\begin{array}{l}32_{18}\\72_{38}\end{array};\ \times\text{ by 3, add 1 or 6, and cast out the tens.}$$

$$\left.\begin{array}{cccccc}&2&4&6&8&0\\&9&5&1&7&3\\\text{or}&4&0&6&2&8\end{array}\right\}\begin{array}{l}32_{68}\\72_{88}\end{array};\ \times\text{ by 3, add 3 or 8, and cast out the tens.}$$

$$\left.\begin{array}{cccccc}&1&3&5&7&9\\&6&2&8&4&0\\\text{or}&1&7&3&9&5\end{array}\right\}52_{28};\ \times\text{ by 3, add 3 or 8, and cast out the tens.}$$

$$\left.\begin{array}{cccccc}&1&3&5&7&9\\&6&0&4&8&2\\\text{or}&1&5&9&3&7\end{array}\right\}04_{34};\ \times\text{ by 2, add 4 or 9, and cast out the tens.}$$

$$\left.\begin{array}{cccccc}&1&3&5&7&9\\&3&7&1&5&9\\\text{or}&8&2&6&0&4\end{array}\right\}\begin{array}{l}04_{84}\\44_{14}\end{array};\ \times\text{ by 2, add 1 or 6, and cast out the tens.}$$

$$\left.\begin{array}{cccccc}&2&4&6&8&0\\&8&2&6&0&4\\\text{or}&3&7&1&5&9\end{array}\right\}24_{24};\ \times\text{ by 2, add 4 or 9, and cast out the tens.}$$

$$\left.\begin{array}{cccccc}&2&4&6&8&0\\&5&9&3&7&1\\\text{or}&0&4&8&2&6\end{array}\right\}24_{74};\ \times\text{ by 2, add 1 or 6, and cast out the tens.}$$

$$\left.\begin{array}{cccccc}&1&3&5&7&9\\&5&9&3&7&1\\\text{or}&0&4&8&2&6\end{array}\right\}\begin{array}{l}44_{64}\\84_{44}\end{array};\ \times\text{ by 2, add 3 or 8, and cast out the tens.}$$

	2	4	6	8	0		
	4	8	2	6	0	64_{04};	× by 2, add 0 or 5, and cast out the tens.
or	9	3	7	1	5		
	2	4	6	8	0		
	6	0	4	8	2	64_{54};	× by 2, add 2 or 7, and cast out the tens.
or	1	5	9	3	7		
	1	3	5	7	9		
	2	6	0	4	8	84_{94},	× by 2, add 0 or 5, and cast out the tens.
or	7	1	5	9	3		
	2	4	6	8	0		
	5	9	3	7	1	16_{06};	× by 2, add 1 or 6, and cast out the tens.
or	0	4	8	2	6		
	2	4	6	8	0	16_{56}	
	7	1	5	9	3	;	× by 2, add 3 or 8, and cast out the tens.
or	2	6	0	4	8	56_{36}	
	1	3	5	7	9		
	6	0	4	8	2	36_{46};	× by 2, add 4 or 9, and cast out the tens.
or	1	5	9	3	7		
	1	3	5	7	9		
	3	7	1	5	9	36_{96};	× by 2, add 1 or 6, and cast out the tens.
or	8	2	6	0	4		
	2	4	6	8	0	56_{86}	
	4	8	2	6	0	;	× by 2, add 0 or 5, and cast out the tens.
or	9	3	7	1	5	96_{16}	
	1	3	5	7	9		
	2	6	0	4	8	76_{26};	× by 2, add 0 or 5, and cast out the tens.
or	7	1	5	9	3		
	1	3	5	7	9		
	4	8	2	6	0	76_{76};	× by 2, add 2 or 7, and cast out the tens.
or	9	3	7	1	5		
	2	4	6	8	0		
	6	0	4	8	2	96_{66};	× by 2, add 2 or 7, and cast out the tens.
or	1	5	9	3	7		
	2	4	6	8	0	08_{02}	
	6	2	8	4	0	;	× by 3, add 0 or 5, and cast out the tens.
or	1	7	3	9	5	88_{42}	
	2	4	6	8	0	08_{52}	
	8	4	0	6	2	48_{22};	× by 3, add 2 or 7, and cast out the tens.
or	3	9	5	1	7	88_{92}	
	1	3	5	7	9	28_{12}	
	7	3	9	5	1	;	× by 3, add 4 or 9, and cast out the tens.
or	2	8	4	0	6	68_{32}	

$$\begin{array}{lccccc} & 1 & 3 & 5 & 7 & 9 \\ & 4 & 0 & 6 & 2 & 8 \\ \text{or} & 9 & 5 & 1 & 7 & 3 \end{array}\Bigg]\begin{array}{l} 28_{62} \\ 68_{82} \end{array}; \times \text{ by 3, add 1 or 6, and cast out the tens.}$$

$$\begin{array}{lccccc} & 2 & 4 & 6 & 8 & 0 \\ & 0 & 6 & 2 & 8 & 4 \\ \text{or} & 5 & 1 & 7 & 3 & 9 \end{array}\Bigg] 48_{72}; \times \text{ by 3, add 4 or 9, and cast out the tens.}$$

Instead of adding a number its complement $(10 - N)$ may be subtracted.

Instead of the odd numbers 1, 3, 5, 7, 9, put O, and instead of the even numbers 2, 4, 6, 8, 0 put E. We have then for cubes ending in 5 and their roots the following facts

$$\begin{array}{cc} 1 & 6 \\ E & O \end{array}\Bigg]\begin{array}{l} 25_{05} \\ 25_{45} \\ 25_{85} \end{array}; \quad \begin{array}{cc} 1 & 6 \\ O & E \end{array}\Bigg]\begin{array}{l} 25_{25} \\ 25_{65} \end{array}; \quad \begin{array}{cc} 3 & 8 \\ E & O \end{array}\Bigg]\begin{array}{l} 75_{15} \\ 75_{55} \\ 75_{95} \end{array}; \quad \begin{array}{cc} 3 & 8 \\ O & E \end{array}\Bigg]\begin{array}{l} 75_{35} \\ 75_{75} \end{array}.$$

It would be trespassing too much upon the limited space of the ANALYST to mention the beautiful harmonies displayed in these facts. I leave them for the reader to observe, hoping that some one may have the time and ingenuity to connect them in a mnemonic formula.

By means of the series preceding 32_{18} I detected an error in the table of cubes in Hutton's Mathematical Dictionary, Vol. 2. He gives 10,360,282 as the cube of 218 instead of 10,360,232.

NOTE ON THE SOUND HEARD IN CONNECTION WITH CONCHS AND OTHER SHELLS, BY GARVIN SHAW, Kemble, Ont., Canada.—When a conch, or any other shell of a similar construction is applied to the ear, a sound is generally heard resembling that of the waves at the sea shore. This phenomenon is produced by the successive reverberations of external sounds against the interior side of the conch. The external sound enters the conch, and is reflected from side to side until it has reached the extrmity of the winding cavity, and the noise which is heard is merely the continued echo.

That this explanation is true is manifest from the circumstance that the intensity of the sound in the conch varies with that of the external sound. If there be no noise whatever connected with the surrounding objects, there will be none in the conch, but if the external noise be great, the internal will also be considerable; nor is this effect connected solely with conchs or other shells of similar construction. If any hollow vessel be applied to the ear while there is an external sound, a continued noise will appear to issue from the vessel. The explanation therefore is true, and that it is also sufficient to explain the phenomenon is obvious.

NOTE ON THE SIGNIFICANCE OF THE SIGNS + AND — BEFORE THE RADICAL $\sqrt{}$.

BY PROF. C. H. JUDSON, FURMAN UNIVERSITY, GREENVILL, S. C.

It is usual with Algebraists to write the roots of the quadratic $x^2 - 2px = q$ in this form; $x = p \pm \sqrt{(q + p^2)}$: whence it is inferred that the sign + before the radical indicates that the *positive root only* is to be taken, whilst the sign — indicates that the *negative root only* is to be taken. This is the express teaching of the ablest Algebraists. See Wood's Algebra by Lund, Cambridge, Eng., page 113—Todhunter's Algebra, page 170—Peacock's Symbolical Algebra, Vol. II, page 106—Hackley's Algebra, N. Y., page 214.

Wood and Todhunter give the following example:

$$x + \sqrt{(5x + 10)} = 8, \ldots\ldots \quad (1)$$

or $\sqrt{(5x + 10)} = 8 - x$; (2) whence $5x + 10 = 64 - 16x + x^2$ or $x^2 - 21x = -54$, whence $x = 18$ or 3. "But," they say, "it appears on trial that 18 does not satisfy the original equation, (1), but it belongs to the equation $x - \sqrt{(5x + 10)} = 8$." Is this doctrine defensible? Let us examine it.

Equation (2) shows that if $x = 3$ then $\sqrt{(5x + 10)} = 8 - 3$, or $+ 5$ and (1) becomes $3 + [5] = 8$, and the equation is *verified.*

If $x = 18$, then (2) gives $\sqrt{(5x + 10)} = 8 - 18 = -10$, and (1) is also verified, thus $18 + [-10] = 8$.

Hence we infer that the + sign before the radical indicates that the true root is to be taken with its proper sign; whilst the — sign indicates that the true root is to be taken with its sign changed.

If the original equation had been $x - \sqrt{(5x + 10)} = 8 \ldots\ldots\ldots (1)$ or $x - 8 = \sqrt{(5x + 10)}$, (2) we should still have had $x^2 - 21x = -54$ and $x = 3$ or 18, as before. But now we have $\sqrt{(5x + 10)} = 3 - 8 = -5$ and (1) is verified; $3 - [-5] = 8$: also $\sqrt{(5x + 10)} = 18 - 8 = 10$, and (1) becomes $18 - [10] = 8$.

This last case shows conclusively that the — sign before the radical does *not* indicate that the negative root only is to be taken, but that the *true* root is to be taken with its sign *changed.*

If we take the equation $x + 5\sqrt{x} = -6$, we have $\sqrt{x} = -\frac{5}{2} \pm \frac{1}{2} = -3$, or -2, and $x = 9$ or 4. But the solution shows that the *negative* root is to be taken in both instances. If we take $x - 5\sqrt{x} = -6$ we have $\sqrt{x} = \frac{5}{2} \pm \frac{1}{2} = 3$ or 2 and $x = 9$ or 4; but now the *positive* roots only are to be taken, notwithstanding the — sign before the radical.

According to the received doctrine, the foregoing equations have absolutely no roots, real or imaginary. And this is maintained by the writers named. Thus also Young (Theory and Solution of Equations of Higher Orders, page 26,) says, "no expression, real or imaginary, can satisfy the equation $2x - 5 + \sqrt{(x^2 - 7)} = 0$." But why should the $+$ sign before the radical indicate that the positive root is taken, any more than the $+$ sign understood before the $2x$ indicates that the positive value only of x is to be taken? The foregoing examples and reasoning appear to be *conclusive*; and if the opposite doctrine—that generally received—can be maintained on any other ground than that of high authority, I should be glad to see it presented in the ANALYST.

A DEMONSTRATION OF THE BINOMIAL THEOREM FOR NEGATIVE EXPONENTS.

BY DAVID TROWBRIDGE A. M., WATERBURGH, NEW YORK.

$$\text{Let } (1+x)^{-\frac{m}{n}} = A_0 + A_1x + A_2x^2 + A_3x^3 + \ldots \ldots \ldots \ldots \quad (1)$$

$$(1+x')^{-\frac{m}{n}} = A_0 + A_1x' + A_2x'^2 + A_3x'^3 + \ldots \ldots \ldots \ldots \quad (2)$$

$$\text{Put} \qquad u^{-m} = (1+x)^{-\frac{m}{n}}, \quad u_1^{-m} = (1+x')^{-\frac{m}{n}},$$

$$\text{then} \quad u^n = 1+x, \quad u_1^n = 1+x', \quad u^n - u_1^n = x - x'. \ldots \ldots \ldots \quad (3)$$

Subtract (2) from (1), then

$$u^{-m} - u_1^{-m} = -\frac{u^m - u_1^m}{u^m u_1^m} = A_1(x - x') + A_2(x^2 - x'^2) + A_3(x^3 - x'^3) \ldots$$

Divide the second member by $u^n - u'^n$, and the third member by $x - x'$, Eq. (3). Then

$$-\frac{u^m - u_1^m}{u^m u_1^m (u^n - u_1^n)} = -\frac{1}{u^m u_1} \left(\frac{u^{m-1} + u^{m-2}u_1 + u^{m-3}u_1^2 + \ldots + u_1^{m-1}}{u^{n-1} + u^{n-2}u_1 + u^{n-3}u_1^2 + \ldots + u_1^{n-1}} \right)$$
$$= A_1 + A_2(x + x') + A_3(x^2 + xx' + x'^2) + \ldots$$

Let $x' = x$, then $u = u'$, and

$$-\frac{mu^{m-1}}{nu^{n-1}u^{2m}} = -\frac{m}{n} \cdot \frac{1}{u^n u^m} = -\frac{m}{n}\Big(1+x\Big)^{-\frac{m}{n}-1} = A_1 + 2A_2x + \ldots$$

Multiply both members by $(1+x)$, and both members of (1) by $m \div n$, then, since $A_0 = 1$,

$$0 = \tfrac{m}{n}(1 + A_1x + A_2x^2 + \ldots + A_px^p + \ldots) + (1+x)[A_1 + 2A_2x +$$
$$3A_3x^2 + \ldots + (p+1)A_{p+1}x^p + \ldots]$$

$\therefore\ \frac{m}{n}A_p + pA_p + (p+1)A_{p+1} = (\frac{m}{n}+p)A_p + (p+1)A_{p+1} = 0;$

and $A_{p+1} = -\frac{\frac{m}{n}+p}{p+1}.A_p$. Put $p=0$, 1, 2, &c., then $A_1 = -\frac{m}{n}$, $A_2 =$

$$-\frac{\frac{m}{n}+1}{2}.A_1 = \frac{\frac{m}{n}\left(\frac{m}{n}+1\right)}{1.2},\ A_3 = -\frac{\frac{m}{u}\left(\frac{m}{n}+1\right)\left(\frac{m}{n}+2\right)}{1\,.\,2\,.\,3}\ \text{\&c.}$$

This demonstration is just as easy as that for positive fractional exponents.

NOTE.—F. W. D. Holbrook, Wakefield, Mass., writes—"Would it be asking too much to request of some of your contributors the equation of a right solid the base of which is an ellipse and the surface of which is generated by a straight line generatrix, moving on the perimeter of the ellipse and also on a straight line parallel to and symmetrical with said ellipse—The solid would be called I presume a conical wedge. The fact is that sections of the solid parallel to its elliptical base are concentric with the cllipse but still are not ellipses.

Or, in other words, can I obtain a demonstration that no two ellipses can be parallel? This inquiry grew out of the designing of a bridge pier of elliptical base where it became necessary to draw diagrams of every course with the size and position of the stone to be used marked thereon.

If the altitude of this wedge were infinite I suppose the base would approach indefinitely near to a circle"

GEOMETRICAL PROBLEM AND SOLUTION.

BY ISAAC H. TURRELL, CUMMINSVILLE, OHIO.

Problem.—A given circle S, touches a straight line at T, and a series of circles Y, Y_1 &c. are drawn tangent to each other consecutively and also touching S and its tangent; then if the ratio of any one of the series as Y to S be represented by a^2, the ratio of Y_n to S will be $(\sqrt{a} \pm n)^4$ according as Y_n is nearer to, or farther from T than Y is. (The positive sign may be regarded as showing the standard case).

As a particular example, if any one of the series becomes equal to S, the square roots of their radii are inversely as the natural numbers 1, 2, 3, 4, &c.

Solution.—Let P, a point in the tangent line, be the external center of similitude of the circles S and Y (radii x, y), then O_1, the center of Y_1 will be found by laying off $PR = PA$, drawing RM perpendicular to PT and equal to the radius y, as in the Figure, joining OM which will bisect RF in V. (F being the point of contact of circle Y and the tangent). A perpendicular to OM at V will meet RM in O_1.

Since q, q_1 are anti-homologous points on S and Y, $Pq \times Pq_1 = PA^2 = PR^2$, $\therefore$ R is the point of tangency of Y_1 and the tangent, and as OO_1M is an isosceles triangle, it is evident that $O_1R = O_1q_1$. A similar construction will show that Y_1 touches S.

Again, by drawing OH parallel to the tangent, and remembering that $TF = 2\sqrt{(xy)}$, it is easily found from the similar triangles HSO, TSP, that

$$PF = \frac{2y\sqrt{(xy)}}{x-y} \text{ and } PT = \frac{2x\sqrt{(xy)}}{x-y};$$

but F, T, being anti homologous points $PF \times PT = PR^2$.

$$\therefore \qquad PR = \frac{2xy}{x-y}, \ RV = \tfrac{1}{2}(PR - PF) = \frac{xy - y\sqrt{(xy)}}{x-y}.$$

In the right triangle O_1VM, $VR^2 = RM \times O_1R$ or $O_1R\ (= y_1)$

$$= \frac{y[x - \sqrt{(xy)}]^2}{(x-x)^2}.$$

If the ratio of S to Y is a^2, $ay = x$ and

$$y_1 = \frac{x}{(\sqrt{a}+1)^2} \text{ or } x = y_1(\sqrt{a}+1)^2.$$

By putting $(\sqrt{a}+1)^2$ for a in the last expression, $x = y_2(\sqrt{a}+2)^2$, and by induction $x = y_n(\sqrt{a}+n)^2$; $\therefore$ $S = Y_n(\sqrt{a}+n)^4$.

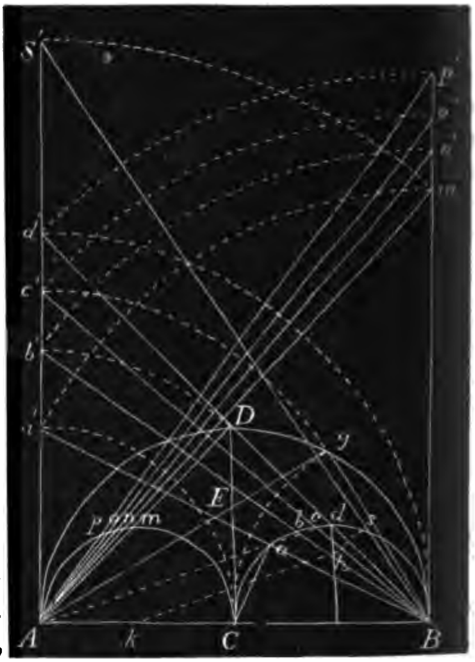
The radii y, y_1 &c. are to each other, inversely as the quantities a, $(\sqrt{a}+1)^2$, $(\sqrt{a}+2)^2$, &c. Hence, if $a = 1$, the square roots of these quantities form the natural series 1, 2, 3, 4, &c.

GEOMETRICAL PROBLEM AND SOLUTION.

BY T. P. STOWELL, ROCHESTER, N. Y.

Problem.—To divide a square into any number of equal squares.—Or to construct a square which shall equal any fractional part of a given square.

Solution.—Let AB be the side of any given square, upon which describe a semicircle, and on each radius AC and CB, describe other semicircles as shown in the diagram.



From B as centre and radius BC describe the arc Cg. Join Ag cutting the radius CD in E. Then, assuming $AB=1$, we have by similar triangles $Ag^2 : AB^2 :: AC^2 : AE^2$ or $1 - \frac{1}{4} : 1 :: \frac{1}{4} : \frac{1}{3}$ Hence the square described on $AE = \frac{1}{3}$ the square on AB.

Again, from A as centre and radius AC describe the arc Ca'—Aa' being perpendicular to AB. Join Ba' cutting the semicircle on CB in a. Then because $Ba'^2 : AB^2 :: CB^2 : Ba^2$, therefore $1 + \frac{1}{4} : 1 :: \frac{1}{4} : \frac{1}{5}$; or the square desribed on $Ba = \frac{1}{5}$ the square on AB.

In a similar manner, $Bb = \frac{1}{6}$, $Bc = \frac{1}{7}$, $Bd = \frac{1}{8}$, $Am = \frac{1}{9}$, $An = \frac{1}{10}$, $Ao = \frac{1}{11}$, $Ap = \frac{1}{12}$ &c., of the square on AB; and there is a regular law governing the construction as shown in the diagram.

Again, suppose we wanted to construct a square $= \frac{7}{18}$ of a given square. From the above construction we know that $Bs^2 = \frac{1}{18}$ of AB^2 and $Bc^2 = \frac{1}{7}$ of AB^2. With radius Bs describe the arc sh cutting Bc in h. Join Ac and draw through h a line parallel with Ac cutting AB in k; then $Bk^2 = \frac{7}{18}$ of AB^2. Because by similar triangles $Bc^2 : BA^2 :: Bh^2 = Bs^2 : Bk^2$, or, multiplying the antecedents by 7, $7Bc^2 : BA^2 :: 7Bh^2 : Bk^2$. But by construction $7Bc^2 = BA^2$; $\therefore 7Bh^2 = Bk^2$, and because $Bh^2 = \frac{1}{18}$ of $BA^2 \therefore Bk^2 = \frac{7}{18}$ of BA^2, and similarly for any other fractional part.

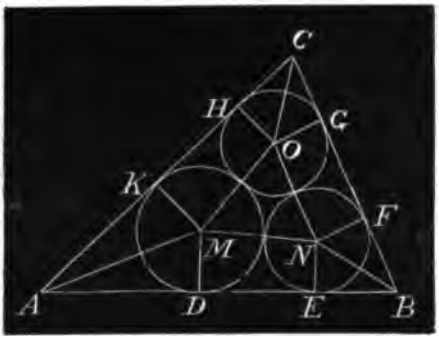
It is evident that a circle, or any regular figure, may be divided in similar fractional parts by the same construction.

SOLUTION OF A PROBLEM.

BY E. B. SEITZ, GREENVILLE, OHIO.

To determine the radii of three circles inscribed in a triangle whose sides are a, b, c, each cir. touching the other two, and also two sides of the triangle.

Solution.—Let ABC be the triangle, M, N, O, the centers of the circles, D, E, F, G, H, K, the points of tangency.



Put $MD=x$, $NE=r_1^2x$, $OG=r_2^2x$, and let $r=$ the radius of the inscribed circle of the triangle. Then $AD=AK=x\cot\frac{1}{2}A$, $BE=BF=r_1^2x\cot\frac{1}{2}B$, $CG=CH=r_2^2x\times\cot\frac{1}{2}C$, $DE=2r_1x$, $HK=2r_2x$, $FG=2r_1r_2x$, and we obtain the following equations.

$$x\cot\tfrac{1}{2}A+2r_1x+r_1^2x\cot\tfrac{1}{2}B=c, \quad\ldots\ldots\ldots\ (1)$$

$$x\cot\tfrac{1}{2}A+2r_2x+r_2^2x\cot\tfrac{1}{2}C=b, \quad\ldots\ldots\ldots\ (2)$$

$$r_1^2x\cot\tfrac{1}{2}B+2r_1r_2x+r_2^2x\cot\tfrac{1}{2}C=a. \quad\ldots\ldots\ldots\ (3)$$

By Trigonometry we have $b(\cot\frac{1}{2}A-\tan\frac{1}{2}B)=c(\cot\frac{1}{2}A-\tan\frac{1}{2}C)$, . (4)

$$ar_1^2(\cot\tfrac{1}{2}B-\tan\tfrac{1}{2}A)=cr_1^2(\cot\tfrac{1}{2}B-\tan\tfrac{1}{2}C), \quad\ldots\ldots (5)$$

$$\frac{\sin\frac{1}{2}B\cos\frac{1}{2}B}{b}=\frac{\sin\frac{1}{2}C\cos\frac{1}{2}C}{c},\ldots (6) \qquad \frac{\sin\frac{1}{2}A\cos\frac{1}{2}A}{a}=\frac{\sin\frac{1}{2}C\cos\frac{1}{2}C}{c}\ldots (7)$$

Dividing (1) by (2) and (3), and clearing of fractions, we have

$$b(\cot\tfrac{1}{2}A+2r_1+r_1^2\cot\tfrac{1}{2}B)=c(\cot\tfrac{1}{2}A+2r_2+r_2^2\cot\tfrac{1}{2}C),\ldots (8)$$

$$a(\cot\tfrac{1}{2}A+2r_1+r_1^2\cot\tfrac{1}{2}B)=c(r_1^2\cot\tfrac{1}{2}B+2r_1r_2+r_2^2\cot\tfrac{1}{2}C). (9)$$

Subtracting (4) from (8) and (5) from (9), we have

$$b(\tan\tfrac{1}{2}B+2r_1+r_1^2\cot\tfrac{1}{2}B)=c(\tan\tfrac{1}{2}C+2r_2+r_2^2\cot\tfrac{1}{2}C),\ldots (10)$$

$$a(\cot\tfrac{1}{2}A+2r_1+r_1^2\tan\tfrac{1}{2}A)=c(r_1^2\tan\tfrac{1}{2}C+2r_1r_2+r_2^2\cot\tfrac{1}{2}C). (11)$$

Multiplying (10) by (6) and (11) by (7), and extracting the square root, we have

$$\sin\tfrac{1}{2}B+r_1\cos\tfrac{1}{2}B=\sin\tfrac{1}{2}C+r_2\cos\tfrac{1}{2}C, \quad\ldots\ldots (12)$$

$$\cos\tfrac{1}{2}A+r_1\sin\tfrac{1}{2}A=r_1\sin\tfrac{1}{2}C+r_2\cos\tfrac{1}{2}C. \quad\ldots\ldots (13)$$

Subtracting (13) from (12), we find

$$r_1=\frac{\sin\frac{1}{2}C+\cos\frac{1}{2}A-\sin\frac{1}{2}B}{\sin\frac{1}{2}C-\sin\frac{1}{2}A+\cos\frac{1}{2}B}=\frac{\sin\frac{1}{2}C+\sin\frac{1}{2}C\cos\frac{1}{2}(2B+C)}{\sin\frac{1}{2}C+\sin\frac{1}{2}C\cos\frac{1}{2}(2A+C)}$$

$$=\frac{\cos\frac{1}{2}C+\cos\frac{1}{2}(2B+C)}{\cos\frac{1}{2}C+\cos\frac{1}{2}(2A+C)}=\frac{\cos\frac{1}{2}B\cos\frac{1}{2}(\pi-A)}{\cos\frac{1}{2}A\cos\frac{1}{2}(\pi-B)}=\frac{1+\tan\frac{1}{2}A}{1+\tan\frac{1}{2}B}.$$

Similarly we find

$$r_2=\frac{\cos\frac{1}{2}C\cos\frac{1}{2}(\pi-A)}{\cos\frac{1}{2}A\cos\frac{1}{2}(\pi-C)}=\frac{1+\tan\frac{1}{2}A}{1+\tan\frac{1}{2}C}$$

From (3) we have $x = \dfrac{a}{r_1^2\cot\frac{1}{2}B + 2r_1r_2 + r_2^2\cot\frac{1}{2}C}$

$$= \frac{a\sin\frac{1}{2}B\sin\frac{1}{2}C}{r_1^2\cos\frac{1}{2}B\sin\frac{1}{2}C + 2r_1r_2\sin\frac{1}{2}B\sin\frac{1}{2}C + r_2^2\sin\frac{1}{2}B\cos\frac{1}{2}C}$$

$$= \frac{r\cos\frac{1}{4}A\cos\frac{1}{4}(\pi - B)\cos\frac{1}{4}(\pi - C)}{2\cos\frac{1}{4}\pi\cos\frac{1}{4}B\cos\frac{1}{4}C\cos\frac{1}{4}(\pi - A)} = \frac{\frac{1}{2}r(1+\tan\frac{1}{4}B)(1+\tan\frac{1}{4}C)}{1+\tan\frac{1}{4}A}.$$

$$\therefore\ r_1^2x = \frac{\frac{1}{2}r(1+\tan\frac{1}{4}A)(1+\tan\frac{1}{4}C)}{1+\tan\frac{1}{4}B},\quad r_2^2x = \frac{\frac{1}{2}r(1+\tan\frac{1}{4}A)(1+\tan\frac{1}{4}B)}{1+\tan\frac{1}{4}C}.$$

SOLUTION OF MR. CHURCH'S PROBLEM.

BY PROF. E. W. HYDE, ITHACA, N. Y.

"Given four points [no one point lying within the triangle formed by the other three] to construct geometrically the axis and focus of the parabola passing through them.

Solution.—Let the four points be a^0, b, c, d^0. Regard them as lying in the horizontal plane of projection, and draw a ground line GL through two of them as b and c. Take some point p [marked $p^h p^v$] in the second angle, and draw the right lines $\overline{pd}$ and $\overline{pa}$ piercing the vertical plane in d_1 and a_1. b and c are in GL and therefore in both the horizontal and vertical planes of projection.

Draw in the vertical plane a horizontal line L^0 through p^v. Now if an ellipse be drawn passing through the four points $a^0{}_1$, b, c, and $d^0{}_1$, and also tangent to the line L^0, and if p be taken as the vertex of the projecting cone, the ellipse $a^0{}_1$ b c $d^0{}_1$ t^0 will be projected into the required parabola, upon the horizontal plane. For by the construction b and c are their own projections, $a^0{}_1$ is projected into a^0, and $d^0{}_1$ into d^0, and t^0 is projected to infinity. It follows therefore that p^h t^h is a diameter of the parabola. It is not necessary to construct the ellipse. Draw by Pascal's theorem a tangent to the ellipse at

either of the given points as b. In the figure $b\,e^{\circ}$ is the tangent. Project it upon the horizontal plane into L°_1, the tangent to the parabola at b. Having now the tangent and diameter at b, other tangents as those at c and c_1 can be drawn at once, and the curve can be constructed, and its axis and focus found.

It only remains to show how to find the point t° at which the ellipse through a°_1, b, c, d°_1 is tangent to L°.

Suppose t° to have been found, and consider $a^{\circ}_1\,b\,c\,d^{\circ}_1\,t^{\circ}$ as an inscribed hexagon with the side at t° reduced to a point; then by Pascal's theorem the intersections of the three pairs of lines $\overline{d^{\circ}_1\,t^{\circ}}$ and $\overline{b\,a^{\circ}_1}$, $\overline{a^{\circ}_1\,t^{\circ}}$ and $\overline{c\,d^{\circ}_1}$, $\overline{b\,c}$ and L° must lie in the same straight line. But as the last two are parallel, the line through the three intersections will de parallel to $\overline{b\,c}$, or GL.

If therefore we draw a series of lines $\overline{\delta_2\epsilon_2}$, $\overline{\delta_3\epsilon_3}$ etc. parallel to GL, and draw also the lines $\overline{d^{\circ}_1\delta_1}$, $\overline{d^{\circ}_1\delta_2}$ etc. and $\overline{a^{\circ}_1\epsilon_1}$, $\overline{a^{\circ}_1\epsilon_2}$ etc., then the locus ot the intersections α, β, γ, etc. of these last lines will cut L° at the required point of contact t°.

This locus is a conic section with its center at k the middle point of $a^{\circ}_1\,d^{\circ}_1$ which is a diameter. It has a pair of conjugate diameters parallel to $c\,d^{\circ}_1$ and $b\,a^{\circ}_1$, and its equation referred to these diameters is

$$\frac{x^2}{\frac{1}{4}a(a\pm a_1)}+\frac{y^2}{\frac{1}{4}b(b\pm b_1)}=1;$$

in which a is the distance from the intersetion of $c\,d^{\circ}_1$ and $b\,a^{\circ}_1$ to the point d°_1; a_1 that from the same intersection to ζ; b that from the intersection to a°_1; and b_1 that from the intersection to η; the lines $a^{\circ}_1\,\zeta$ and $\eta\,d^{\circ}_1$ being parallel to GL.

If the upper signs of a_1 and b_1 be taken the curve is an ellipse, if the lower, it is an hyperbola. The first corresponds to the case where the lines $\delta_1\,\epsilon_1$, $\delta_2\,\epsilon_2$ etc. are drawn across the other angle between the lines $\overline{cd^{\circ}_1}$ and $\overline{ba^{\circ}_1}$, i. e. so as to cut one of them on the same side of their intersection on which a°_1 and d°_1 are situated, and the other on the other side, while the second is the case in the figure. If a°_1 and d°_1 are equally distant from GL, or if $\overline{ba^{\circ}_1}$ and $\overline{cd^{\circ}_1}$ are parallel the locus reduces to two intersecting right lines.

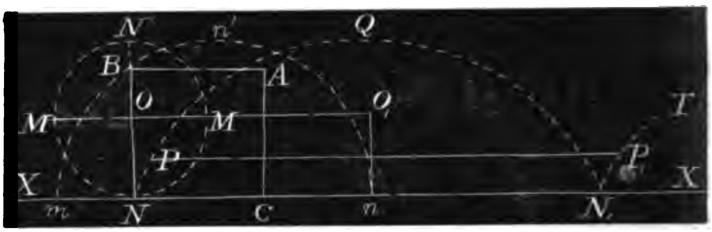
QUADRATURE OF THE CIRCLE.

BY PROF. AUGUST ZIELINSKI, AUGUSTA, GEORGIA.

The Geometers of Antiquity used to distinguish, in the solution of mathematical problems, the geometrical solutions and the mechanical ones—the

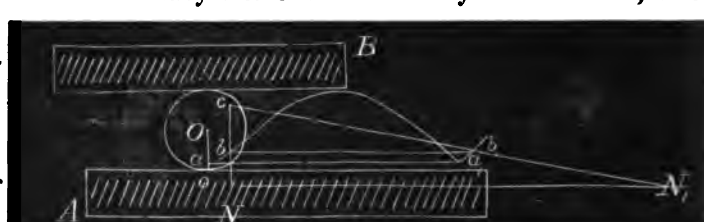
former being executed by means of the ruler and the dividers only, the latter by any other more complicated means.

The solution of the problem of the quadrature of the circle which I am going to give is a mechanical one.



If $NMN'M'$ is the given circle and we let it roll on a straight line XX' which is a tangent to the circumference at N, each point of the circumference will describe a cycloid, and the bases of all these cycloids will be equal. Let NQN' be the cycloid described by the point N of the circumference, and $N'T$ the next cycloid described by this point; if PP' is parallel to XX', then $NN' = PP'$. But, as N_1 is the point on the tangent XX' where the point N of the circumference comes when the circle $NMN'M'$ has made a complete revolution, we have, if $r = ON$, the radius of the generating circle, $NN' = 2r\pi$. If we make $Nn = \frac{1}{2}NN_1 = r\pi$, we will have rectangle $ONnO_1 = r^2\pi$. If we make $Nm = On = r$, and on mn as diameter describe the semi-circle $mn'n$, it will cut the perpendicular NN' in B, and we shall have $mN : NB :: NB : Nn$, whence $(NB)^2 = Nn \,.\, mN$; that is, $a^2 = (NB)^2 = r \,.\, r\pi = r^2\pi$ or $a = NB$ is one side of a square $NBAC$, whose area is equal to that of the given circle $NMN'M'$.

If we adapt to a small heavy disc O of brass or anyother material, whose radius R is known, a pencil a, and putting it on a plane with the point of the pencil



downward, we apply a ruler A to its circumference; if then by means of another ruler B pressed against the opposite point of its circumference, and moved quickly in a horizontal direction, parallel with the stationary ruler, we produce a rotation and progressive motion of the disc, the point of the pencil a, will draw on the plane a cycloid aba_1 whose base will be parallel to the edge of the ruler; then, if bb_1 is parallel aa_1, $bb_1 = 2R\pi$.

If $cb = R = Oo$, and we want the circumference of a circle of radius $R_1 = cN$, we draw the straight line cN_1 through b_1 making NN_1 parallel to bb_1, then is $cb : cN :: bb_1 : NN_1$, or $R : R_1 :: 2R\pi : NN_1$; $\therefore$ $NV_1 = 2R_1\pi$.

Hence by means of a single cycloid we can transform any circ. into a square.

RECENT MATHEMATICAL PUBLICATIONS.

COMMUNICATED BY G. W. HILL.

ANNALES DE L'OBSERVATOIRE DE PARIS, PUBLIÉES PAR U. J. LE VERRIER, Directeur de l'Observatoire. Mémoires. Tome X. GAUTHIER-VILLARS. Paris, 1874. 4to. 422pp. 27 fr.

In this volume the author treats the mutual perturbations of Jupiter and Saturn, employing the same method as that which in previous volumes he has applied to the theories of the inferior planets, viz: the method of variation of the elements. A whole volume has not sufficed to complete the subject. At the end is a memoir by M. M. Wolf and André, "On the singular appearances which have frequently accompanied the observation of the contacts of Mercury and Venus with the Sun's limb."

APERÇU HISTORIQUE SUR L'ORIGINE ET LE DÉVELOPEMENT DES MÉTHODS EN GÉOMÉTRIE, PARTICULIÈREMENT DE CELLES QUI SE RAPPORTENT À LA GÉOMÉTRIE MODERNE, SUIVI D'UN MÉMOIRE DE GÉOMÉTRIE SUR DEUX PRINCIPES GÉNÉRAUX DE LA SCIENCE. LA DUALITÉ ET L'HOMOGRAPHIE. 2d Édition, conforme à la première. PAR. M. CHASLES. GAUTHIER-VILLARS. Paris. 1875. 4to. 850pp. Price for subscribers, 25 fr.

The house of GAUTHIER-VILLARS has undertaken to print this rare and much sought-after volume. It is to be hoped that they will continue the good work, and put again within the reach of students the equally rare GEOMETRIE SUPERIEURE of the same author.

JOURNAL DE L'ÉCOLE POLYTECHNIQUE. 44th Cahier. GAUTHIER-VILLARS. Paris, 1874. 4to.

This number contains six memoirs by M. Maximilien Marie on the elementary theory of integrals, single, double, and of any order, and on their residuals and periods, together with an extension of Cauchy's method to double integrals, and a classification of the integrals which express the quadrature of algebraic curves; a note by M. Badoureau on the problem of resolutions applied to chess and similar games; a new determination of the velocity of light, by M. Cornu; and a memoir by M. Hermite on certain definite integrals.

ESSAI SUR UNE MANIÈRE DE REPRÉSENTER LES QUANTITÉS IMAGINAIRES DANS LES CONSTRUCTIONS GÉOMÉTRIQUES. PAR R. ARGAND. 2d Édition précédée d'une Preface par M. HOUËL. GAUTHIER-VILLARS. Paris, 1874. 8vo. 5 fr.

EXPOSITION DE LA MÉTHODE DES ÉQUIPOLLENCES, PAR G. BELLAVITIS. Traduit de l'italien par C. A. LAISANT. GAUTHIER-VILLARS. Paris, 1874. 8vo. $4\frac{1}{2}$ fr.

THÉORIE DES FONCTIONS DES VARIABLES IMAGINAIRES. Tome I. Nouvelle Géométrie Analytique ou Extension des méthodes de la Géométrie de Descartes à l'étude des courbes et surfaces qui peuvent être représentées par les solutions imaginaires des équations à deux ou trois variables. PAR MAXIMILIEN MARIE. GAUTHIER-VILLARS. Paris, 1874. 8vo.

EXPOSITION ÉLÉMENTAIRE DES DIVERSES THÉORIES DE LA GÉOMÉTRIE MODERNE. PAR M. LENTHERIC. GAUTHIER-VILLARS. Paris, 1874. 4to. 6½ fr.

SUR LES SURFACES TRAJECTOIRES DES POINTS D'UNE FIGURE DE FORME INVARIABLE DONT LE DÉPLACEMENT EST ASSUJETTI À QUATRE CONDITIONS. PAR A. MANNHEIM. GAUTHIER-VILLARS. Paris, 1874. 4to. 1½ fr.

EXPOSITION DE LA MÉTHODE DE HANSEN RELATIVE AU CALCUL DES PERTURBATIONS DES PETITES PLANÈTES. PAR L. DUPUY. Paris, 1874. 8vo.

THEORIE DER PLANETEN-VORÜBERGÄNGES VOR DER SONNENSCHEIBE. VON DR. KARL FRIESACH. Engelmann. Leipzig, 1874. 8vo. ½ th.

ATLAS DES SÜDLICHEN GESTIRUTEN HIMMELS. Darstellung der zwischen dem südpol und dem 20 Grad südlicher Abweichung mit blossen Augen sichtbaren Sterne nach ihren wahren, unmittelbar von Himmel entnommenen Grössen. Nebst einem Stern-Verzeichnisse. VON DR. CARL BEHRMANN. Brockhaus. Leipzig, 1874. 8vo. and oblong 4to. 3½ th.

BEOBACHTUNGEN DER SONNENFLECKEN ZU ANCLAM. VON PROF. G. SPÖRER (Publication der Astronomischen Gesellschaft.) Leipzig, 1874. 5 th.

ASTRONOMICAL AND METEOROLOGICAL OBSERVATIONS MADE DURING THE YEAR 1872, at the U. S. Naval Observatory. Washington, 1874. 4to.

ASTRONOMICAL AND METEOROLOGICAL OBSERVATIONS MADE DURING THE YEAR 1872, at the Royal Observatory of Greenwich. London, 1874. 4to.

A NEW THEORY OF THE RULE OF SIGNS.

BY LEVI W. MEECH A. M., HARTFORD, CONN.

Prof. De Morgan has remarked that Sturm's Theorem "is the complete *theoretical* solution of a difficulty upon which energies of every order have been employed since the time of Descartes." But the amount of labor which it often requires, has turned back the attention of Algebraists to shorter methods. Accordingly the design of the present article will be to fix upon one central principle, easy of recollection and application, strictly demonstrated, and approaching progressively toward the completeness of Sturm's Theorem by the shortest modes of inspection. A first advance of mine in the investigation of a particular case of this kind was inserted in the Mathematical Monthly for 1858.

I. In every equation, wrote Descartes, there may be as many positive roots as there are changes of sign or passages from the sign $+$ to the sign $-$, or the contrary; and as many negative roots as there are successions of the same sign. In this equation for example, $x^3 - 17x^2 + 79x - 63 = 0$, there are three changes of sign, and the three roots are positive, namely 1, 7, 9. Let us multiply it by $x + 4$; we shall have this result $x^4 - 13x^3 + 11x^2 + 253x - 252 = 0$; where there are actually three changes of sign, which indicate the three positive roots, with a succession of the same sign by reason of the negative root.

The general statement, being left by Descartes incomplete and without demonstration, encountered much opposition, according to Montucla, from whose history the above extract is translated; but improvments have followed slowly and surely. In entering this field of investigation, let us first consider a new and important extension.

II. *Order of signs and magnitude of the Roots.* In multiplying the first equation from Descartes by $x + 4$, the inquiry arose in the mind of the writer, where will the new permanence enter among the three former variations? Denoting a permanence of sign by p, and a variation of sign by v, it was interesting to observe, it had located in the order $vvpv$ corresponding with the roots 9, 7, -4, 1, arranged in descending order of magnitude.

Applying this clue to equations of the second degree, it was readily proved that the right hand variation or permanence defines the sign of the least root, and the next defines the sign of the greatest root. For illustration, the quadratic $x^2 - 4x - 21 = 0$, indicates vp; and the corresponding roots are 7, -3, in descending magnitude, as before. Again take the biquadratic equation $x^4 + 17.658x^3 - 65.341x^2 + 36.5087x + 2.1576 = 0$.

Scale. *pvvp*, or — + + —. Now the four roots are actually, $x = -20.87$, $+2.50$, $+0.768$, -0.0538. All of which correspond with the scale both in sign and in order of magnitude.

From induction of numerous resolved examples, this *double correspondence*, suggested as above described, is generally true. As such it will be presently indicated in the new Rule. In the only exception yet found, the two roots $+.76$, $-.73$, border upon equal roots, where transposition is admissible.

Before proceeding further, let us multiply the first equation from Descartes by a factor having imaginary roots, as by $x^2 + 20x + 110$; the product will be, $$x^6 + 3x^4 - 151x^3 - 353x^2 + 7430x - 6930 = 0.$$

Scale. *pvpvv*, or — + — + +.

It should be observed that the two imaginary roots are accounted negative; they do not enter adjacent to each other, but are alike. If we had multiplied by $x^2 + 2x + 10$, the product would have given the scale + + + + +; here the two imaginary roots enter as positive, but still alike Therefore the factor representing two imaginary roots should generally have the form of $x^2 \pm ax + b$, where b is positive.

By way of corollary, in the case of equal roots, with or without contrary signs, the corresponding signs of the scale may be indifferently transposed, since there is no advance in value. And in the only other instance known at present, of an inverted order, the two roots $+.764$, $-.732$ were *nearly* equal. Thus with the occasional exception of roots approximately equal, the double correspondence of the scale has so far been found entirely applicable.

III. *Demonstration of Descartes' rule.* The common method consists chiefly in proving that the introduction of one more positive root into an equation, is always accompanied by at least one more variation of sign. Thus let a denote any positive root, to be introduced into the following equation:

(1) $$x^{m-1} + Ax^{m-2} + Bx^{m-3} + \ldots Tx + U = 0.$$

Multiplying by $x - a$ to introduce the root a, we have the equation,

(2) $$x^m \begin{array}{l|} + A \\ - a \end{array} x^{m-1} \begin{array}{l|} + B \\ - aA \end{array} x^{m-2} \begin{array}{l|} + C \\ - aB \end{array} x^{m-3} \ldots \begin{array}{l|} + U \\ - aT \end{array} x \; - aU = 0.$$

When each term containing x in equation (2) has the same sign as its upper letter, the order of signs will evidently be the same as in (1), with the gain of one variation at the end, where $-aU$ has a sign contrary to that of the preceding term determined by U.

But more generally the equation (2) commences on the left, with a compartment of one or more terms having the same signs as in (1) or the upper letters; then comes a reverse compartment of terms with signs opposite to those of (1), and which consequently give the same permutations as in (1),

and so on alternately; thus,

(2) Original sign or signs. Reverse. Original. Reverse. . . . Reverse.

(2) $O\ R\ O\ R\ O\ R\ O\ R\ O \ldots R.$

Let $n + 1$ denote the number of left hand junctions of the reverse with original signs, in the order ($O\ R$), at each of which a variation is always gained; then will n evidently denote the number of right hand junctions in the order ($R\ O$), at each of which a variation may be either lost or gained; which results are thus proved:

Let A and $(B - aA)$ respectively denote the signs of the last coefficient of an original and the first of the next reverse compartment. In order that the part or product $- a \times A$ may have a sign opposite to that of B, as well as exceed B, it is evident that A and B must have like signs, and so give a permanence in (1), while the corresponding terms in (2) give a variation. Thus at the left hand junction of every reverse compartment ($O\ R$), a variation is gained.

Again let $- C$ and $(D - aC)$ respectively denote the signs of the last of a reverse, and the first of the next original compartment. At this right hand junction, the signs in (1) will evidently be those of C, D; the signs in (2) will be those of $- C$, D. If the former denote a permanence, the latter will denote a variation, and conversely.

Thus at the $n + 1$ left hand junctions of reverse compartments, $n + 1$ variations are always gained in (2) over (1), while at the n right hand junctions only n variations can be lost. Hence the introduction of a positive root is always attended by the net gain of at least one variation in (2) over the previous number in (1).

The nature of the results may also be exhibited as follows:

$$\left.\begin{array}{l}\text{At } n + 1 \text{ left hand}\\ \text{junctions, } (O\ R)\ldots\end{array}\right\}\begin{array}{l}\text{in (1) } A,\ B, \text{ permanence only.}\\ \text{in (2) } A,\ -B, \text{ variation only.}\end{array}$$

$$\left.\begin{array}{l}\text{At } n \text{ right hand}\\ \text{junctions, } (R\ O)\ldots\end{array}\right\}\begin{array}{l}\text{in (1) } C,\ D, \text{ variation or permanence.}\\ \text{in (2) } -C,\ D, \text{ permanence or variation.}\end{array}$$

The result is entirely conclusive, that the number of real positive roots cannot exceed the variations of sign. And in like manner, the multiplication by $x + b$ to introduce a negative root, would necessarily give at least one more permanence of signs; whence a similar conclusion follows.

In the next place, let us suppose the signs of the first and last terms of equation (1) to be alike; then must the number of variations along its terms, be zero or an even number; while the first and last terms of (2) being then unlike, the number of variations in (2) must be an odd number. Again, let us suppose the first and last terms of (1) to have unlike signs, which must indicate in total, an odd number of variations; while the first and last terms

of (2) being then alike, must indicate an even number in (2). Thus in every case, the gain of variations in (2) over the previous number in (1), being the difference between an odd and an even number, is always an odd number

IV. *Incomplete Equations.* We are at liberty to supply the place of deficient terms with zero terms or infinitesimals, having such signs as to render the number of variations a minimum; and so of the permanences. Now if the inserted zeros have the same sign as the preceding term, as in $+ x^5 + 0 + 0 — x^2$, they can evidently give no new variation; and their insertion is superfluous, in respect to *positive* roots.

In respect to *negative* roots, an even number of zero terms whose signs alternate with the preceding, as in $— x^5 + 0 — 0 + x^2$, can give no new permanence; since the last zero has the same sign as the preceding real term Consequently the insertion of zero terms is superfluous, when the difference of the including exponents is an *odd* number, or the number of vacant terms is even.

But when the difference is an *even* number, between the two including exponents, let us *first* suppose the two including signs to be unlike, as in $+ 3x^6 — 0 + 0 — 0 — 5x^2$. Here the alternating signs admit of no permanence except at the junction of the last zero with the next real term; which invariably gives one permanence. As the terms $+ 3x^6 — 5x^2$ have previously indicated one *variation*, and now are made by the inserted zero to indicate one permanence, the total result will be rendered by simply copying both the given signs + and —, into the scale of roots, as described in the new Rule.

Lastly, the difference of the two including exponents being still an *even* number, let us *secondly* suppose the signs of the two including terms to be alike, as in $+ x^4 — 0 + 2x^2$. In this, and all such cases, the alternate signs admit not of any permanence. And the previous like signs gave no variation. Hence the two corresponding roots are not real but imaginary; and the correct result will be rendered by temporarily omitting either of the like signs, or by remembering to regard the succession in $+ x^4 + 2x^2$, and such terms, as a permanence of imaginary roots only, and omitting it from the scale of the new Rule.

Thus the effect of zero or vacant terms can be generalized into the Rule, so that their further insertion is no longer needed. Nor does it appear necessary here to give separate precepts for the corresponding imaginary roots; since the number of signs or roots of the scale, positive and negative, subtracted from the degree of the equation will give the even excess of imaginary roots, beyond any pair or pairs of real roots noted in the scale.

The invention ot Fluxions or the Calculus has opened the way to the next simple criterion of imaginary roots; to which let us now proceed.

V. *The Quadratic Type.* This name is here suggested for any three terms of an equation, where the three exponents are in the series of natural numbers, and the first and third terms, or extremes, have like signs; whatever may be the sign of the mean or middle term; such as, $-13x^2+10x-49$, or $Px^5 \pm Qx^4 + Rx^3$.

The leading idea is, that, when the final differential coefficient, placed equal to zero, has imaginay roots, the original equation must have at least an equal number of them. Let it here be observed, that the types to give independent indications of imaginary roots, can only be contiguous; that is, the middle term of one must not occur in the next, but the last term of one type may be taken as the first of the next type. Let it also be remembered, to guard against a common mistake, that if all the types in an aquation give real roots, we cannot infer conversely that the roots of the original equation are all real; for imaginary roots are still possible, with certain exceptions.

Let the original equation be denoted by,

$$u = Ax^m + \ldots + Px^n + Qx^{n-1} + Rx^{n-2} + \ldots = 0.$$

Differentiating $n-2$ times, we then substitute the reciprocal of y in place of x. Multiplying by y^{m-n+2}, and differentiating $m-n$ times with respect to y, then restoring the value of y, and omitting common factors, we obtain the isolated type,

$$n(n-1)Px^2 + 2(n-1)(m-n+1)Qx + (m-n+2)(m-n+1)R = 0.$$

Resolving this quadratic,

$$x = \frac{m-n+1}{nP}\Big[-Q \pm \sqrt{(Q^2 - fPR)}\Big],$$

where

$$f = \frac{(m-n+2)n}{(m-n+1)(n-1)}.$$

If the quantity under the radical sign be negative, the two roots of the last derivative are imaginary, and consequently two roots of the original equation are imaginary. This is rendered very evident by tracing the curve of any equation and its derivatives, after the manner of Rolle, of which some elegant examples are given in Montucla.

It will be noticed that the value of f is the ratio of two consecutive factors of the general binomial coefficient. When the degree of the equation m is an odd number, the middle or minimum value of f is

$\frac{m+3}{m-1}$, being always greater than unity. And when m is an even number, the middle or minimum value of f is $\left(\frac{m+2}{m}\right)^2$. Equi-distant from the middle, the values of f are equal to each other. Thus,

When the quadratic type begins with the first term, or ends with the last term of an equation, we have $n=m$, and $f=\frac{2m}{1(m-1)}$.

When the type begins with the regular second term, or ends with the usual last term but one, then will $n=m-1$, and $f=\frac{3(m-1)}{2(m-2)}$.

When the type begins with the normal third term, counted in from either extremity of an equation, $n=m-2$, and $f=\frac{4(m-2)}{3(m-3)}$.

To recapitulate,—In the common *quadratic* equation two imaginary roots exist when Q^2-4PR is negative.

In any equation of the *third* degree, two roots are imaginary, if Q^2-3PR is negative.

In any equation of the *fourth* degree, the negative type of two imaginary roots, is $Q^2-\frac{8}{3}PR$; that is, if P denotes the first term or R the last term of the equation. And in the middle part the type is $Q^2-\frac{9}{4}PR$.

In any equation of the *fifth* degree the values of f are, $f=\frac{5}{2}, 2, \frac{5}{2}$. Yet the general formula may be preferable; thus, *to find f*;—

The highest exponent of the type is one factor of f, and the difference between the lowest exponent of the type and the degree of the whole equation, is the other factor of the numerator. Subtracting unity from each, gives the two factors of the denominator.

The quadratic type occurs, for example, in the last three terms of,

$$x^4-x^3-13x^2+16x-48=0.$$

Here $f=\frac{8}{3}$, and $Q^2-fPR=(16)^2-\frac{8}{3}\times 13\times 48=-1408$. This result being negative, it follows that two roots are imaginary.

The quadratic type occurs in the first three terms of the equation,

$$x^3+3x^2+7x+4=0.$$

Here $3^2-3\times 1\times 7$ being negative, two roots are imaginary.

It may be remarked that if the first and last terms of the type had unlike signs, no calculation were needed to show that since $-PR$ is then positive, the criterion would be positive and indecisive.

VI. *The Cubic Type.* By this term we designate any three or four terms of an equation, which terms divided by their lowest power of x would give the form of the common cubic equation. For example,

$$+Px^n+Qx^{n-1}+Rx^{n-2}+Sx^{n-3}.$$

Let us suppose the given equation to be first divided by P, and let $Q\div P=Q'$, etc., then,

$$A'x^m+\ldots+x^n+Q'x^{n-1}+R'x^{n-2}+S'x^{n-3}+\ldots=0.$$

Differentiating $n-3$ times, and dividing by $(n-3)\ldots 3.2.1$,

$$\left(\ldots\right)x^{m-n+3}\ldots+\frac{n(n-1)(n-2)}{1.2.3}x^3+\frac{(n-1)(n-2)}{1.2}Q'x^2+\frac{n-2}{1}R'x$$
$$+S'=0.$$

Dividing through by x^{m-n+3}, then substituting y in place of the reciprocal of x, differentiating $m-n$ times with respect to y, then restoring x and dividing by the left hand coefficient,

$$x^3+\frac{3(m-n+1)}{n}Q'x^2+\frac{3(m-n+1)(m-n+2)}{n(n-1)}R'x$$
$$+\frac{(m-n+1)\ldots(m-n+3)}{n(n-1)(n-2)}S'=0.$$

If this final derivative has two imaginary roots, to be ascertained presently by solution or by Sturm's theorem, then must the preceding derivative, and consequently the original equation, have two such roots, as previously described. Causing the second term to disappear by making

$$x=\frac{(m-n+1)}{n}\left(z-Q'\right),$$

$$z^3+3(fR'-Q'^2)z+2Q'^3-3fQ'R'+f\frac{(m-n+3)n}{(m-n+1)(n-2)}.S'=0.$$

The general equation $z^3+qz+r=0$, is proved in Algebra to have two imaginary roots whenever $-(\frac{1}{3}q)^3-(\frac{1}{2}r)^2$ is negative. Substituting and dividing by Q'^6, which cannot alter the sign, the criterion becomes,

$$\left(1-\frac{fR'}{Q'^2}\right)^3-\left(1-\frac{3fR'}{2Q'^2}+\frac{f(m-n+3)n}{2(m-n+1)(n-2)}\cdot\frac{S'}{Q'^3}\right)^2=neg.$$

Or multiplying and dividing the last term by $(fR'\div Q'^2)^2$ denoted by f'^2, and introducing g' to be presently described, the criterion is,

$$(1-f')^3-(1-\tfrac{3}{2}f'+\tfrac{1}{2}f'^2g')^2=negative.$$

Expanding and multiplying by $4\div f'^2$, which cannot alter the sign, the criterion becomes symmetrical; thus,

$$(C)\qquad 3-4(f'+g')+6f'g'-(f'g')^2=neg.$$

Now to facilitate the practical application, let us suppose this criterion to be the ordinate, and g' the abscissa of a continuous curve. When g' is infinite positive or negative, the criterion is evidently negative in both branches of the curve; while the intermediate portion between the two roots of g', is positive. Thus when the criterion is placed equal to zero, the two roots g_1, g_2 will be limits; outside of, and not between which, the actual value of g' must fall to indicate two imaginary roots. We find,

$$g_1=\frac{3f'-2-2(1-f')^{\frac{3}{2}}}{f'^2}.$$

$$g_2 = \frac{3f' - 2 + 2(1-f')^{\frac{3}{2}}}{f'^2},$$

$$= \tfrac{3}{4} + \tfrac{1}{8}f' + \tfrac{3}{64}f'^2 + \tfrac{3}{128}f'^3 + \ldots.$$

These formulas will next be tabulated; the degree of the given equation being m; the cubic type, $Px^n + Qx^{n-1} + Rx^{n-2} + Sx^{n-3}$,

$$f' = \frac{n(m-n+2)}{(n-1)(m-n+1)} \cdot \frac{PR}{Q^2}, \qquad g' = \frac{n-1}{n-2} \cdot \frac{m-n+3}{m-n+2} \cdot \frac{QS}{R^2}.$$

Here the first factors are formed alike from the exponents of P, R and Q, S, by the simple precept before given under the quadratic type.

OUTLINE TABLE OF THE LIMITS.

f'	g_1	g_2	f'	g_1	g_2	f'	g_1	g_2
+1.00*	+1.000	+1.000	0.0	—∞	+0.750	—2.000	—4.598	+0.598
0.90	+0.786	0.942	—0.1	—460.7	0.738	—3.000	—3.000	0.556
0.80	+0.345	0.904	—0.2	—130.7	0.726	—5.000	—1.855	0.496
0.75	0.000	0.889	—0.3	—65.16	0.716	—7.000	—1.385	0.446
0.70	—0.467	0.875	—0.4	—40.71	0.706	—9.000	—1.139	0.423
0.60	—1.961	0.850	—0.5	—28.70	0.697	—20.00	—0.636	0.326
0.50	—4.828	0.828	—0.6	—21.80	0.688	—40.00	—0.404	0.252
0.40	—10.81	0.810	—0.7	—17.41	0.680	—60.00	—0.315	0.214
0.30	—25.24	0.793	—0,8	—14.42	0.672	—80.00	—0.266	0.190
0.20	—70.78	0.778	—0.9	—12.27	0.664	—100.0	—0.233	0.173
+0.10	—340.7	+0.766	—1.0	—10.66	+0.657	—1000.	—0.066	+0.060

In respect to vacant terms, if either P or S is zero then f' or g' is zero; and by the table, the other g' or f' must equal or exceed 0.75 to indicate two imaginary roots. That is, simply, $0.75 - f' = negative$.

When $Q = 0$; we multiply the criterion by Q^2R^2 and then make $Q = 0$, reducing the criterion to $-4PR^3 - f(gPS)^2$. When P and R have *like* signs, this is negative, so that two roots are imaginary, as under the quadratic type. When P and R have *unlike* signs, $-PR$ is positive, and the criterion is,

$$(C') \qquad -\frac{R^3}{P} - \frac{fg^2}{4} . S^2 = neg., \text{ where } g = \frac{n-1}{n-2} \cdot \frac{m-n+3}{m-n+2}.$$

In like manner, when $R = 0$, the criterion is,

$$(C'') \qquad -\frac{Q^3}{S} - \frac{gf^2}{4} . P^2 = negative.$$

(*To be continued.*)

*This first value indicates generally two imaginary roots, otherwise two equal roots.

PHOTOGRAPHIC OBSERVATIONS OF THE TRANSIT OF VENUS.

BY PROF. ASAPH HALL, NAVAL OBSERVATORY, WASHINGTON, D. C.

EDITOR ANALYST:

I was so busy during my absence, besides being cut off from the mail, that I failed to keep my promise to write you. Let me try to make amends now by giving you a short account of the photographic operations.

The plan adopted by the American Commission for making photographs of the Transit of Venus is now pretty well understood. It consists simply of a lens of 38 to 40 feet focal length, so placed that its axis lies in the meridian, and is made horizontal by means of a spirit level. This lens is mounted on an iron pier firmly set in the ground, and south of this lens, at its focus, is fixed the plate holder, also mounted on an iron pier. The plate holder carries the sensitive plate, and directly in front of this plate a plumb line of fine silver wire. In front of the plumb line is firmly fixed to the plate holder a well made plate of clear glass, on which are finely etched two sets of straight lines, the lines being half an inch apart and the sets at right angles to each other. The distance from the north surface of the ruled glass plate to the surface of the sensitive plate is half an inch. On every photograph we have therefore traces of the ruled or etched lines and of the plumb line. North of the lens is placed a heliostat which moves a plane glass mirror so that the sun's rays are reflected through the lens to the sensitive plate. The whole apparatus is simple, it is firmly mounted, and it gives the means of determining accurately the polar coordinates of a point on the photograph with respect to an assumed point.

First we have a photograph of the sun, very nearly a circle four inches in diameter, and we note and record the instant of local time when the slide passes and admits the sun's rays and this photograph is made. For this instant we can compute for the known latitude of the station the position o the sun in the heavens, and hence the angular distance from the bottom or top of the photograph to the north point of the sun's image. The trace of the plumb line furnishes a known direction to which the measurement of the angles may be referred. If therefore we assume a point as the sun's center we can determine by measuring the photograph with a position micrometer the position angle of any other point, as the center of venus, with respect to the plumb line, or to a declination circle passing from the pole of the heavens through the center of the sun. This position angle can be found, I think, with much greater accuracy than it can be with an equatorial telescope, and here is one advantage of the American method. Secondly we must have

the means of converting any distance, say an inch, on the photograph very exactly into arc; so that if we measure the distance between the two points and convert it in to arc we have the second polar coordinate. This conversion can be made if we know the distance from the surface of the lens to the surface of the sensitive plate, just as we find the angular value of a micrometer screw by measuring the focal distance of the objective and counting the number of threads of the screw to an inch. The distance between the surfaces has been measured with an accuracy that seems much within the limits required. We have therefore the second polar coordinate and the solution is theoretically complete.

But in order to establish the usefulness of the photographic method it remains to be shown that the photographs can be made in distant places transported thousands of miles through great change of temperature, that the collodion film is subject to no contraction or change except such as can be fully accounted for in the measurement and calculation, and finally that their linear measurement can be made with the required accuracy. We may easily get some idea of the accuracy required in the measurements of the photographs. The sun's diameter is about 32 minutes of arc, and the photographs beeng 4 inches in diameter we shall have $\frac{1}{1000}$ of an inch equal to $0''.48$, and $\frac{1}{10000}$ of an iuch equal to $0''.05$. The photographs to be measured are the negatives on the glass plates, and on these Venus appears as a round vacant spot $\frac{1}{8}$ of an inch in diameter. As this spot is a symmetrical one, and generally well defined it is probable that the pointings on this spot can be made with sufficient accuracy. The difficulty will be in fixing the position of the sun's center. This can be done only by proceeding from the edge of the photograph, and the edge, or limb, of the sun is a very difficult thing to deal with, on account of its inequalities. Here perhaps some assistance may be derived from the systems of ruled lines, which are designed to control any contraction of the collodion film, and which may be used as a system of rectangular coordinates to which the centers of the sun and Venus may be referred.

We have now at the Naval Observatory a number of photographs from different stations, and shall probably soon know with what accuracy they can be measured.

SOLUTIONS OF PROBLEMS.

Solutions of problems in No. 2 have been received as follows:
From J. M. Arnold, 63; A. L. Baker, 61 & 65; W. W. Beman, 59, 60,

61, 63, 65 & 66; Marcus Baker, 59, 60, 61, 65, 66 & 67; Prof. P. E. Chase, 59, 60, 61 & 65; Geo. M. Day, 59, 60, 61, 63, 64, & 65; Dr. H. Eggers, 62 & 63; E. S. Farrow, 59, 60, 61, 63, 64, 65, 66 & 67; Henry Gunder, 59, 60, 61, 63, 64, 65 & 66; Wm. Hoover, 59, 60, 61, 63, 65 & 66; G. W. Hill, 67; Artemas Martin, 59, 60, 61, 63, 64, 65 & 66; J. B. Mott, 61; F. P. Matz, 59 & 60; E. P. Norton, 63; O. D. Oathout, 59, 60 & 65; L. Regan, 59, 60, 61, 62 & 63; E. B. Seitz, 59, 60, 61, 63, 64 & 65; Walter Siverly, 59, 60, 61, 62 & 65; E. T. T., 59 & 60; J. S. Mays, 59 to 66.

59.—"Find a number, consisting of two places of figures, whose half squared will equal the number inverted."

SOLUTION AY PRES. E. T. TAPPAN, KENYON COLLEGE, GAMBIER, OHIO.

Problem 59 gives this equation $[\frac{1}{2}(10x + y)]^2 = 10y + x$, $\therefore y = 20 - 10x \pm \sqrt{(400 - 396)}$. Evidently the only integral value of x is 1. Then $y = 8$.

SOLUTION BY PROF. PLINY EARLE CHASE, PHILADELPHIA, PA.

Let $2x$ be the number sought. Then, in accordance with the Arabic system of numeration, $x^2 - 2x = x(x-2) = 9y$. Since x, the larger factor, can contain only one figure, if $2x$ is an integer y must be the smaller factor. $\therefore x = 9$; $y = 7$; $2x = 18$. If $2x$ is decimal, its right-hand digit must be 0, and the square of half the tenths digit must be one tenth of the digit. $\therefore x^2 = .2x$; $x = .20$; $y = -1.8$; $2x = .40$.

[Because the same digits that represent $2x$ also represent x^2; therefore the excess of 9's contained in the sum of the digits is the same in both cases; consequently the excess of 9's in $x^2 - 2x$ is zero. $\therefore x^2 - 2x$ is a multiple of 9, equal $9y$ say.—Ed.]

60.—"Find a value for x that will make $\frac{62x + 1984}{x^2 - 1024}$ a whole number."

SOLUTION BY PRES. E. T. TAPPAN.

The given expression reduces to $\frac{62}{x - 32}$, which is a whole number when $x - 32$ is a measure of 62. Hence x may be 94, 63, $52\frac{2}{3}$, $47\frac{1}{2}$, &c.; 35, 33 &c.

61.—"What will be the value of each letter of the alphabet if the product of all but a is 1, all but b is 2, all bnt c is 3, and so on to all but z is 26."

SOLUTION BY PROF. P. E. CHASE.

Let $p =$ the product. Then

$$\frac{p}{a} = \frac{p}{2b} = \frac{p}{3c} = \ldots \frac{p}{26z}. \quad p = a = 2b = 3c = \ldots 26z = \frac{a^{26}}{2.3.4\ldots 26}.$$

$$a = \sqrt[25]{(2.3.4.\ldots.26)} = 11.59375;\ b = 5.79688;\ c = 3.86459;\ \&c.$$

62.—"Given four lines in a plane: To inscribe a parallelogram within them with given direction of sides."

SOLUTION BY DR. H. EGGERS, MILWAUKEE, WISCONSIN.

The four given right lines may be named a, b, c, d, and the two given directions p and q.

Take on the line a an arbitrary point P, draw from P a line parallel to p, till it intersects line b in Q; from Q draw a line parallel to q, intersecting line c in R; from R draw a line parallel to p, intersecting line d in S; from S draw a line parallel to q intersecting line a in a point M. Now if the points P and M should coincide, the problem would be solved. But as this circumstance in general will not occur, take on the line a another arbitrary point P' and proceed exactly as above, till a second point M' results on line a, analogous to point M: Now take a third arbitrary point P'' on line a and proceeding in the same manner, construct a third point M'' on line a. The two systems of points on line a, viz; P, P', P'' and M, M', M'' will form two homographic systems of points on line a. Find now the two coincident points of these two systems of points, which can be done by one fixed arbitrary circle and the ruler alone. These two points may be X and Y; then every one will furnish one solution of the problem. For we have only to repeat the above trial construction, beginning with point X or Y, instead of the arbitrary point P; then the fourth line connecting d with a will pass through point X or Y respectively.

If The succession of the four lines a, b, c, d, is fixed, the number of solutions will be in general two. But if all possible combinations of the four given lines are admissible, we find 12 different combinations, consequently $12 \times 2 = 24$ solutions.

[Mr Siverly's solution of this question is by Analytical Geometry, and is brief and elegant. Mr. J. S. Hays, of Hodgenville, Ky., sent a very ingenious and elegant geometrical costruction and demonstration; his diagram is larger however than we can, consistently, admit to our pages; and we exact of correspondents, when diagrams are necessary to illustrate their subjects, that they be correctly drawn, suitable for the engraver to copy, and not to exceed three inches in their largest dimension; and always less when practicable.]

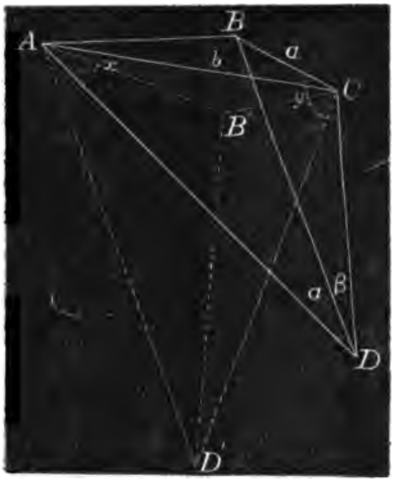
63.—Given the sides AB, AC and BC, and the angles α and β, (see diagram below), to find the sides AD, BD and CD.

SOLUTION BY W. W. BEMAN, UNIVERSITY OF MICH., ANN ARBOR, MICH.

Conctruction. Construct the triangle ABC. Upon AB construct a segment containing an angle of 24°, and upon BC a segment containing an angle of 16°. The intersection of these arcs will determine the point D, the distance of which from A, B and C can now be measured.

It is evident that there will be two solutions, one when B and D are on the same side of AC, and another when they are on opposite sides.

Trigonometrical Computation. Designate the angles and sides as in the figure.



Then $\alpha+\beta+x+y+B=360°$, or $x+y=360°-(\alpha+\beta+B)$. From ABD, $BD=\dfrac{c\sin x}{\sin\alpha}$; from BDC, $BD=\dfrac{a\sin y}{\sin\beta}$. $\therefore\ \dfrac{c\sin x}{\sin\alpha}=\dfrac{a\sin y}{\sin\beta}$, or

$$\frac{\sin x}{\sin y}=\frac{a\sin\alpha}{c\sin\beta}=\tan\phi\ \text{(say)}.$$

Then
$$\frac{\sin x+\sin y}{\sin x-\sin y}=\frac{\tan\frac{1}{2}(x+y)}{\tan\frac{1}{2}(x-y)}$$
$$=\frac{\tan\phi+1}{\tan\phi-1}=\tan(\phi+\tfrac{1}{4}\pi).$$

Hence we can find x and y.

$$BD=\frac{c\sin x}{\sin\alpha}=18.88,\ AD=\frac{c\sin ABD}{\sin\alpha}=23.65,\ CD=\frac{a\sin DBC}{\sin\beta}$$
$=15.17$ ms.

When B and D are on the same side of AC, $x+y=B-(\alpha+\beta)$, but the remainder of the work is the same.

This method of computation is known as Delambre's.

64.—"Find the maximum value of $\left(\dfrac{a}{x}\right)^x$ without the aid of the Calculus".

SOLUTION BY ARTEMAS MARTIN, ERIE, PA.

The given function will be a maximum when its Napierian logarithm is a maximum. $\therefore\ u=x\log a-x\log x$,

and $$u'=(x+h)\log a-(x+h)\log(x+h)$$
$$=(x+h)\log a-(x+h)\log x-\left(x+h\right)\left(\frac{h}{x}\right)+\frac{1}{2}\left(x+h\right)\left(\frac{h}{x}\right)^2-\&c.$$

The coefficient of h is $\log a-\log x-1$, which put $=0$ and we have $x=\dfrac{a}{e}$; $\therefore$ the maximum value of the given function is $e^{\frac{a}{e}}$.

[This question may be solved by a simpler process as follows:

The given expression in logarithms is $x \log a - x \log x =$ a max., $= x \log m$. Dividing by x, $\log a - \log x = \log m$; whence $x = a \div m$, and the original expression becomes $m^{\frac{a}{m}} =$ a max., or $a.\dfrac{\log m}{m} =$ a max. But $\dfrac{\log m}{m}$ is a maximum when m is the base of the system. Therefore $m = e$, and the required maximum is $a.\dfrac{\log e}{e} = e^{\frac{a}{e}}$. —Ed.]

65.— "The corner of a page is turned down, and in every position the area of the triangle is two square inches; find the locus of the angular point."

SOLUTION BY WALTER SIVERLY, OIL CITY, PA.

Let r and θ be the polar coordinates of the locus referred to the corner before it is turned down and one edge of the leaf. The double area of the triangle $= \frac{1}{2}r(\tan\theta + \cot\theta) = 4$, or $r^2 = 16\sin\theta\cos\theta = 8\sin 2\theta$ is the polar equation of the curve, which is therefore the lemniscate.

66.—"*A* speaks the truth b times out of a; B, d times out of c; and C, n times out of m.

C says that B told him that A said a certain event transpired. Required the probability that the event occurred."

[Artemas Martin gets for answer to this question,

$$\frac{bdn}{acm} + \frac{b}{a}\left(1 - \frac{d}{c}\right)\left(1 - \frac{n}{m}\right) + \frac{d}{c}\left(1 - \frac{b}{a}\right)\left(1 - \frac{n}{m}\right) + \frac{n}{m}\left(1 - \frac{b}{a}\right)\left(1 - \frac{d}{c}\right);$$

and for authority he refers to Todhunter's History of the Theory of Probability, p. 462; and Todhunter's Algebra, 4th edition, p. 464. E. S. Farrow gets the same result as Mr. Martin. All the other correspondents give for the required chance $bdn \div acm$.

If we assume that all the witnesses did actually testify *in relation to the event*, then we think there is no doubt that Mr Martin's result is correct:

For, let it be assumed that the event did in fact occur; and suppose, in a given statement, that C tells the truth, and that B, in informing C, told a lie. Then B must have told C that A said the event occurred, therefore, because B lied, A must have told B that the event did not occur; but by hypothesis the event *did* occur, therefore A lied. Hence if C asserts that an event has occurred, and if his statement is based on the fact that B has testified to him in relation to the event, whose testimony in turn is based on A's statement in relation to the event; then it follows that, if any two in making their statements tell a lie and the other one tells the truth, the event must have occurred.

We contend, however, that the question, as announced, does not warrant the inference that all three necessarily testified in relation to the event; and that when C says that B said that A said a certain thing, if C lies, he may either lie in relation to the *character* of the statement which he reports, or he may lie in the affirmation that B *made* a statement; and there is no reason assigned from which we can infer that he would be more likely to lie in respect to the character of the statement than in relation to the fact of the statement. But if we admit that C lied in attributing a statement to B, when no statement of any kind in relation to the event was made by B; and that B in like manner lied in attributing a statement to A, when no statement was made by A; then it is clear that the chance in favor of the event resulting from the concurrence of C and B in a lie with A in the truth, is only one fourth what is assigned to that contingency above And, for the same reason, the chance resulting from the concurrence of C and A, and B and A respectively in telling a lie, with the other one in telling the truth, is one half what is assigned to that contingency above. To warrant Mr. Martin's result we claim therefore that the statement of the question must be modified.

Mr. Martin has appended to his solution of 66, a note correcting a misprint in Todhunter's History of the Theory of Probability, which we give below.—Ed.]

Note, by Artemas Martin. — The denominator of the formula for *traditional* testimony given in *Todhunter's History of the Theory of Probability*, p. 462, third paragraph, is wrong — a misprint probably. I will quote a portion of the second and third paragraphs:

"Suppose a witness to speak truth m times and falsehood n times out of $m + n$ times; let m' and n' have similar meanings for a second witness. Then if they agree in an assertion the probability of its truth is

$$\frac{mm'}{mm' + nn'}. \quad * \ * \ * \ *$$

Using the same notation as before if one witness reports a statement from the report of another the probability of its truth is

$$\frac{mm' + nn.}{(m + m')(n + n')};$$

for the statement is true if they both tell the truth or if they both tell a falsehood."—The latter formula should be

$$\frac{mm' + nn'}{(m + n)(m' + n')}.$$

[We are compelled, for want of room, to defer publishing the solution of problem 67 until the issue of the July No.]

PROBLEMS.

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69. BY E. P. NORTON, ALLEN, MICH.—Given $x^2 + y^2 = 793, \ldots (1)$
$\sqrt[3]{(xy^2)} + \sqrt[3]{(x^2y)} = 30, \ldots . (2)$
to find x and y by quadratics.

70. BY F. P. MATZ, B. E., ED. MATH. DEP. "NATIONAL EDUCATOR", KUTZTOWN, PA.—Given $y^2 + z^2 = 2500, \ldots$ (1), $x^2 = [130 - (x + t)]^2 \times 1600, \ldots$ (2), $xy = t[130 - (x + t)], \ldots$ (3), $t^2 = y^2 + (z - 40)^2 . .$ (4) to find x, y and z.

71. BY WILLIAM HOOVER, BELLEFONTAINE, O.—Given the sides a, b, c of a spherical triangle ABC to find the radii, R, r, of the circumscribed and inscribed circles.

72. BY A. W. MASON, CEDAR FALLS, IOWA.—Find the maximum cylinder that can be cut from a given oblate spheroid, whose semi-axes are a and b.

73. BY CHRISTINE LADD, CHELSEA, MASS.—Find the whole number of sets of three integers having a constant sum.

74. BY PROF. W. W. HENDRICKSON, U. S. NAVAL ACAD. ANNAPOLIS, MD.—Find the equation of the locus of the middle point of a chord to the hyperbola $x^2 - y^2 = 2a^2$, the chord being of constant length and equal to seven times the transverse axis.

QUERY, BY PROF. A. HALL. — Into how many parts can n planes divide space.

QUERY, BY E. T. T. GAMBIER, O.—Is 65537 ($= 2^{16} + 1$) a prime?

Errata. On page 8, line 16, for "$3a^2$" which occurs in the numerator of the fraction, read a^3; and on page 58, line 4, and lines 4 and 6 from bottom, for "l^2" in the numerator of the left member of the equation, read l. Also, on page 59, line 10, multiply the left member of the equation by $\sin\theta$.

[*Note.* Several correspondents have found fault with Mr. Farrow's solution of 53, but no other solution has been offered.

It is evident that the formula used by Mr. Farrow to estimate the effect of the atmospheric resistance, does not apply in calculating the velocity generated while the ball is passing from the breech to the muzzle of the gun, which is the case to be considered; for during that time the only pressure on the rear of the ball is that of the gas generated. The result obtained by Mr. Farrow cannot, therefore, be correct.]

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NOTICE.

The Editor will be absent from home during the month of July, hence, any correspondence received after the 4th of July, cannot receive attention till about the first of August.

THE ANALYST.

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A NEW THEORY OF THE RULE OF SIGNS.

BY LEVI W. MEECH A. M., HARTFORD, CONN.

(*Continued from page* 88.)

For all equations of the third degree, the cubic type is decisive; if it does not indicate imaginary roots, there are none; as appears from comparison with Sturm's theorem. For the fourth and higher degrees, it is decisive only so far as it indicates two imaginary or equal roots; leaving the question of more of them, yet to be decided.

Example. $$x^3 + 2x^2 - 23x - 70 = 0.$$

$$f' = -3 \times 23 \div 2^2 = -17.25, \quad g' = -3 \times 2 \times 70 \div 23^2 = -.794.$$

Since g' and f' are interchangeable, we enter the foregoing Table with -0.8 nearest value of g', and there find that f' must negatively exceed -14.42, which it does, being -17.25, and indicating two imaginary roots.

Example. $$x^3 - 5x^2 + 2x + 12 = 0.$$

$$f' = +6 \div 5^2 = .24, \quad g' = -180 \div 2^2 = -45.$$

Entering the table with $f' = 0.20$ and interpolating for .04, we find for $f' = 0.24$, $g_1 = -52.56$. As g' is *within* this limit, the roots are *all real*, and accord with the Rule of signs. The result is confirmed by computing the criterion (C) with the respective values of f' and g'; and the comparative smallness of (C), being $+0.6$, indicates two roots to be nearly equal.

VII. *The Biquadratic Type*, or five consecutive terms:

$$x^m + \ldots [Px^n + Qx^{n-1} + Rx^{n-2} + Sx^{n-3} + Tx^{n-4}] + \ldots = 0.$$

Differentiating $n-4$ times and proceeding as with former types, also making $m - n = m'$, the final derivative takes the form of

$$x^4 + \frac{4(m'+1)}{n}Q'x^3 + \frac{6(m'+1)(m'+2)}{n(n-1)}R'x^2 + \frac{4(m'+1)\ldots(m'+3)}{n\ldots(n-2)}S'x$$
$$+ \frac{(m'+1)\ldots(m'+4)}{n\ldots(n-3)}T' = 0.$$

Or making $x = \frac{m'+1}{n}(z - Q')$, and reducing,

$$z^4 - 6Q'^2(1-f')z^2 + 8Q'^3(1 - \tfrac{3}{2}f' + \tfrac{1}{2}f'^2g')z + Q'^4(-3 + 6f' - 4f'^2g' + f'^3g'^2h') = 0.$$

Here $Q' = Q \div P$; $h = \dfrac{n-2}{n-3} \cdot \dfrac{m'+4}{m'+3}$, and $h' = \dfrac{hRT}{S^2}$.

For convenience, we may name the similar quantities, f', g', h',.... *triple ratios.*

A decisive criterion is now given by Descartes' or Euler's well known solution of the equation $z^4 + bz^2 + cz + d = 0$, by means of the auxiliary cubic, $y^3 + 2by^2 + (b^2 - 4d)y - c^2 = 0$. This cubic, by substituting $y = y' - \frac{2}{3}b$, becomes, $\quad y'^3 - (\frac{1}{3}b + 4d)y' - \frac{2}{27}b^3 + \frac{8}{3}bd - c^2 = 0.$

The well known criterion of $y'^3 + qy' + r = 0$, being $-(\frac{1}{3}q)^3 - (\frac{1}{2}r)^2 =$ a negative quantity, we substitute from the above equation of z^4 and find the criterion of two imaginary roots in the cubic of Euler's solution to be, after reduction,

$$(1 - \tfrac{4}{3}g' + \tfrac{1}{3}f'g'^2h')^3 - [1 - 2g' + g'^2(f' + h' - f'h')]^2 = neg. \quad \text{Or,}$$

$$(R^2 - \tfrac{4}{3}gQS + \tfrac{1}{3}fg^2hPT)^3 - [(R^2 - gQS)^2 - g^2(Q^2 - fPR)(S^2 - hRT)]^2.R^2 = negative.$$

When the middle coefficient R is zero, the criterion is simply,

$$\tfrac{1}{4}fghPT - QS = neg.$$

And when the given equation is a biquadratic, $\frac{1}{4}fgh$ is 4.

When the second coefficient Q is zero, we can at once apply the cubic type to the auxiliary cubic of Euler's solution above.

Thus, $\qquad f' = \dfrac{3}{4}\left(\dfrac{b^2 - 4d}{b^2}\right) = \dfrac{3}{4} - \dfrac{3d}{b^2};\ g' = -\dfrac{c^2}{(\frac{2}{3}b)^3 f'^2}.$

And the foregoing Table fo Limits will now determine the presence or the absence of two imaginary roots in the cubic. As shown in Algebra, the roots of the biquadratic will be all real, all imaginary, or only two imaginary, according as the cubic equation has three positive, two negative, or two imaginary roots, respectively.

VIII. *General Principle and Rule of Signs.* From the nature of the preceding demonstrations, it is evident that they are but particular cases under the following general principle:

The consecutive portions of a given equation, including contiguous terms, may form minor equations to be placed equal to zero; and each or its derivative, resolved so far as to find the signs or nature of its roots. The total number of imaginary roots so found, will be an inferior limit, and the total number of positive and of negative real roots will be superior limits to the respective numbers of roots in the original equation.

Example. $x^7 - 2x^5 + 2x^3 - 2x^2 + 2x - 1 = 0.$

Assume $x_1{}^7 - 2x_1{}^5 + 2x_1{}^3 = 0$, $2x_2{}^3 - 2x_2{}^2 = 0$, $-2x_3{}^2 + 2x_3 - 1 = 0.$

$$x_1 = \pm\surd[1 \pm\surd(-1)],\ x_2 = +\ ,\ x_3 = \tfrac{1}{2} \pm\surd(-\tfrac{1}{4}).$$

Thus of the seven roots, six must be imaginary, and one positive. The general precept evidently admits the application of all compatible rules of solution or of previous differentiation. But in general practice, it will be most convenient to suppose the equation resolved first into binomials then into trinomials, cubic types, and so on, and the completed results given by inspection according to the following

General Rule. Where the decrement is an odd number as 1, 3, *or* 5, *from one exponent to the next, if there be* NO *change from the former sign to the latter, write the* NEGATIVE *sign; but otherwise, the positive sign, in the scale of roots.*

Where the decrement is an even number, as 2, 4, 6, *from one exponent to the next, compare the two signs, and if there be* NO *change make* NO *entry, but otherwise enter both the unlike signs* + — *in the scale of roots.*

The number of positive roots in the scale, descending by 2, *if possible, will be the alternative number of real positive roots in the given equation. And so with the number of negative roots. An odd number here proves the certain existence of one root of the species.*

Note 1. The same scale of signs generally exhibits the detailed order of magnitude of the roots, descending from the greatest to the least, in the same way with the exponents of the equation.

Note 2. If the preceding method of quadratic type indicates two imaginary roots in any part of the equation, make out the two corresponding signs of the scale of roots. That is, if $1 - f'$ be negative; also if $\frac{3}{4} - f'$ be negative, when the type has a vacant term on either side, according to the normal terms of the equation.

Note 3. Sometimes the first and third coefficient of the quadratic type, having like signs, are numerically equal to, or greater than, the middle coefficient; a pair of imaginary roots is then indicated. And to prepare for this comparison, the first and third coefficient may be, the one multiplied, and the other divided by the same assumed number,

Note 4. If a cubic type within the equation indicates two imaginary roots, mark out two corresponding signs of the scale, which are alike, leaving one sign contrary to that of the last term of the type.

The signs corresponding to a Biquadratic type may be similarly adjusted.

Example. $x^8 - 2x^5 + x^2 - 0.2 = 0.$

Scale, + + + —; that is, one negative, and one or three positive roots.

Example. $x^8 - 1.12x^7 + 2.54x^6 - 2.85x^5 - 7.37x^4 + 4.02x^3 - 0.65x^2$
$+ 2.47x - .0387 = 0.$

By inspection with Note 3, the first three terms form one quadratic type, and the 6th 7th and 8th terms form another independent type, each indicating a pair of imaginary roots, or four in all. The remaining terms by the Rule give the scale $+ - + +$; and the corresponding real roots, found by trial, are $+ 1.65$, $- 1.25$, $+ 0.83$, $+ 0.015$. The attention of Algebraists is again invited to the double correspondence of the scale with the signs, and with the order of magnitude of the real roots.

It may be useful for future reference, to observe briefly, that the criteria of the preceding types, when placed equal to zero, coincide with the equations of condition, otherwise found, for two equal roots. And what appears more remarkable, the quantity $1 - f'$ is identical in sign with the first coefficient in the third function of Sturm's Theorem, and with the sum of the squares of the roots of the given equation, first determined by Newton; it occurs essentially in his rule of signs in a dependent order of similar elements, which might also have been taken independently.

When the ratios f, g, ... in a complete equation are all positive, the Quadratic type at once excludes each pair of consecutive ratios, where one of them exceeds 1, as pertaining to imaginary roots; and the Cubic type excludes most of the remaining pairs, where the greater triple ratio lies between 0.75 and 1. Preparatory to such considerations, we may employ the well known transformation into the equation of the squares of the roots, where all permanences of sign indicate so many imaginary roots, which will aid the complete examination. Or we may substitute $l + z$, or $- l + z$ in place of the unknown x; where l denotes the greatest limit of positive or negatve roots; and the former would render all the roots z negative, and the latter, positive. If the greatest coefficient of the equation be denoted by m, then l may be any convenient number greater than $m + 1$. In either result, the triple ratios f', g', ... will be all positive.

In the particular case of an equation having only one variation of signs, or one positive root, our method of sub-equations will give its limits. The equation is first separated into two or more portions, placed equal to zero, having the first part or highest power of x positive, and only one variation of signs. The greatest and least of the positive roots so found, will be limits, between which lies the true positive root. And their mean will be its approximate value. For example, $x^3 + 3x^2 - 6x - 8 = 0$.

Let $x_1{}^3 - 6x_1 = 0$, $3x_2{}^2 - 8 = 0$; hence $x_1 = 2.45$, $x_2 = 1.53$; between these limits is found the true positive root 2. The mean of the limits is 1.99.

For a second example, $x^4 - 3x^2 - 4x - 3 = 0$.

Let $\frac{1}{3}x_1{}^4 - 3x_1{}^2 = 0$; $\frac{1}{3}x_2{}^4 - 4x_2 = 0$; $\frac{1}{3}x_3{}^4 - 3 = 0$;

$$x_1 = 3,\ x_2 = 2.290,\ x_3 = 1.732.$$

The mean of these is 2.34; the true positive root is 2.302. In these instances, the results have not differed very widely from each other.

APPENDIX. A NEW METHOD OF APPROXIMATION.—The present contribution has extended to greater length than was first contemplated, but an important extension suggested since the commencement of writing, may still merit attention.

I. *Preliminary.* Let us premise that the double correspondence discovered in our rule of signs, will be useful in guiding the choice of the greatest, least, or special root to be found. For illustration, to commence on the right hand side of the equation, $ax^3 - bx^2 - cx - d = 0$, as in reversion of series, or in Lagrange's Theorem, would give the least root, but it is negative. To divide through first by ax^3, so as to commence on the other side, would lead to the determination of the reciprocal of the greatest root, which by inversion is the root itself. The scale of signs indicates the latter course, if the positive root is to be found.

Having chosen which root of the scale is to be computed, we next inquire which of the following approximations will be the shortest.

If the last two terms of the equation placed equal to zero, would give a small fraction of unity as the root, the approximation below for the least root will probably be rapid and easy.

If the first two terms of the equation, placed equal to zero, would give a root materially greater than unity, the process below for the greatest root will probably be the shortest. For the only positive root or the only negative root, or the two least roots or the two greatest roots, or two imaginary roots, either of the last methods can be applied, according to circumstances.

II. *The Least Root* Having transposed all the terms to one side, and divided through by the known term, let r denote the least root, and $\ldots p,\ q$ the other roots. then by the well known property of equations,

$$1 + bx + cx^2 + dx^3 + \ldots . = A(x - r)(x - q)(x - p) \ldots . = 0,$$

$$= A(x - r) \div \frac{1}{(x - q)(x - p)\ldots} = 0.$$

Developing each reciprocal factor in series,

$$\frac{1}{x - q} = -\frac{1}{q} - \frac{xr}{q^2} - \frac{x^2}{q^3} - \ldots$$

When x represents the least root r, this series will evidently converge. Hence the divisor of $A(x-r)$ will be the product of convergent series and itself converge the more rapidly as r is relatively smaller than the other roots. The form and convergence of the divisor being known, we accordingly assume,

$$\frac{1-b'x}{1+c'x+d'x^2+e'x^3+f'x^4+\ldots} = 1+bx+cx^2+dx^3+\ldots=0.$$

When b' is thus determined, the equation $1-b'x=0$, as above shown, will give the least root. Clearing of fractions, and equating the coefficients of equal pawers af x,

$$0=b+b'+c', \qquad x=\frac{1}{b'}=\frac{-1}{b+c'}.$$
$$0=c+bc'+d',$$
$$0=d+cc'+bd'+e',$$
$$\ldots\ldots\ldots\ldots$$

In successive approximations, suppose first that $c'x$ and the following terms are omitted, then $b'=-b$. Next suppose that only $d'x^2$ and the following terms are omitted; then $b'=-b+c\div b$. Next suppose that only $e'x^3$ and following terms are omitted; and so on, to determine by elimination, the value of b' more and more accurately. Let $x_1, x_2, x_3, x_4, \ldots$ denote the corresponding values approaching the true value of x; also let $D_1, D_2, D_3, \ldots$ denote the corresponding denominators:

$$D_1=b; \qquad x_1=\frac{-1}{b}.$$
$$D_2=bD_1-c; \qquad x_2=\frac{-D_1}{D_2}=\frac{-b}{b^2-c}.$$
$$D_3=bD_2-cD_1+d; \qquad x_3=\frac{-D_2}{D_3}=\frac{-b^2+c}{b^3-2bc+d}.$$
$$D_4=bD_3-cD_2+dD_1-e; \qquad x_4=\frac{-D_3}{D_4}.$$
$$D_5=bD_4-cD_3+dD_2-eD_1+f; \quad x_5=\frac{-D_4}{D_5}. \quad \text{etc.}$$

With remarkable simplicity, the values of D are computed one from another after the manner of recurring series; and with changed sign, the ratio of two consecutive values of D gives the approximate value of x. Such is the approximation for the least root.

III. *The Greatest Root.* The preceding analysis shows we have only to commence on the other side of the equation, divided by the term containing the highest power of x. The derivation of D will be the same as before, and for x we must take the reciprocal of the former value. Thus, given the equation, $$x^m+bx^{m-1}+cx^{m-2}+\ldots=0.$$

$$D_1 = b;\ D_2 = bD_1 - c;\ \text{etc., as above.}$$

$$x_1 = -b,\ x_2 = -b + \frac{c}{b},\ x_3 = -b + \frac{cb - d}{b^2 - c},$$

$$x_4 = -b + \frac{c(b^2 - c) - db + e}{b(b^2 - c) - cb + d},$$

$$x_2 = \frac{-D_2}{D_1},\ x_3 = \frac{-D_3}{D_2},\ x_4 = \frac{-D_4}{D_3}, \ldots.$$

Example. Find the greatest or only positive root of the equation,

$$2x^3 - 473x^2 - 234x - 711 = 0,$$

dividing by 2 $\qquad x^3 - \underset{b}{236.5}x^2 - \underset{c}{117}x - \underset{d}{355.5} = 0.$

Here $x_1 = -b = +236.5$, $x_2 = -b + c \div b = 237.00003$. The exact root found by the longer process of Horner is 237.

When a vacancy occurs after the first term, the equation may still be adapted to this method, by transposing all terms after the first, expanding the proper root of both sides, and transposing again.

When the number of vacant terms is large, as in the noted problem solved by Baily, De Morgan and others, for finding the rate of interest of an annuity, (Jones on annuities, p. 42), successive substitution is often the shortest method. The formula referred to is,

$$x = \frac{1}{p} - \frac{1}{p(1+x)^n}.\quad \text{Let } x_1 = \frac{1}{p},\ x_2 = \frac{1}{p} - \frac{1}{p(1+x_1)^n}, \text{ etc.}$$

If $n = 98$ and $p = 21.924788$; $x_1 = .04561047$, $x_2 = .045034$, $x_3 = .0450017$, etc. The true root is .045.

IV. *The two Least Roots, or two Imaginary Roots.* Recurring to the analysis of article II, let another term be added to the assumed number, making it $1 - b'x - b''x^2$. Equating coefficients, we find the conditional equations the same as before, except the second, which is now $0 = c + b'' + bc' + d'$. Thus changed, the first two equations give b', b'', in terms of c', d'; which latter are found from the remaining equations, by the aid of modern determinants. Thus b', b'', being known, the two least roots, real or imaginary, are found by the quadratic,

$$1 - b'x - b''x^2 = 0;\ x = -\frac{b'}{2b''} \pm \sqrt{\left[\left(\frac{b'}{2b''}\right)^2 + \frac{1}{b''}\right]}.$$

$$b' = -b - c', \qquad\qquad b'' = -c - bc' - d'.$$

$$\begin{vmatrix} c & b & 1 & 0 & . & . \\ d & c & b & 1 & . & . \\ e & d & c & b & . & . \\ f & e & d & c & . & . \\ . & . & . & . & . & . \end{vmatrix} c' = -\begin{Vmatrix} d & b & 1 & 0 & . & . \\ e & c & b & 1 & . & . \\ f & d & c & b & . & . \\ g & e & d & c & . & . \\ . & . & . & . & . & . \end{Vmatrix} \qquad \begin{vmatrix} c & b & 1 & 0 & . & . \\ d & c & b & 1 & . & . \\ e & d & c & b & . & . \\ f & e & d & c & . & . \\ . & . & . & . & . & . \end{vmatrix} d' = +\begin{vmatrix} d & c & 1 & 0 & . & . \\ e & d & b & 1 & . & . \\ f & e & c & b & . & . \\ g & f & d & c & . & . \\ . & . & . & . & . & . \end{vmatrix}$$

$$c'_1=0,\ d'_1=0;\ c'_2=-\frac{d}{c},\ d'_2=0;\ c'_3=\frac{be-cd}{c^2-bd},\ d'_3=\frac{d^2-ec}{c^2-bd};..$$

The process will be materially simplified by any deficient terms, as when $d=0$.

To find *the two greatest roots*, we have only to divide by the term containing the highest power of x so as to bring the equation into the following form, and then apply the last process:

$$1+b\left(\frac{1}{x}\right)+c\left(\frac{1}{x^2}\right)+\ldots.=0.$$

THE APPLICATION OF THE EXPONENTIAL POLYGON TO THE HESSIAN.

BY PROF. H. T. EDDY, UNIV. OF CINCINNATI, CINCINNATI, OHIO.

1°. The second differential coefficient of any curve whose equation is

$$u=\psi(x,y)=0$$

is

$$\frac{d^2y}{dx^2}=-\frac{\left(\frac{d^2u}{dx^2}\right)\left(\frac{du}{dy}\right)^2-2\left(\frac{d^2u}{dxdy}\right)\left(\frac{du}{dx}\right)\left(\frac{du}{dy}\right)+\left(\frac{d^2u}{dy^2}\right)\left(\frac{du}{dx}\right)^2}{\left(\frac{du}{dy}\right)^3}\ldots.(a)$$

(Todhunter's Differential Calculus, art. 180). The vanishing of (a) is the condition that the first differential coefficient is a maximum or a minimum. i. e. that the point of tangency is a point of inflection.

The condition then is

$$\left(\frac{d^2u}{dx^2}\right)\left(\frac{du}{dy}\right)^2-2\left(\frac{d^2u}{dxdy}\right)\left(\frac{du}{dx}\right)\left(\frac{du}{dy}\right)+\left(\frac{d^2u}{dy^2}\right)\left(\frac{du}{dx}\right)^2=0.\ldots.(b)$$

which is a function of x and y and is the equation of the curve called the Hessian, which may be written $H=0$. The Hessian passes through all points of inflection of $u=0$.

2°. At a double point of the curve $u=0$, the first differential coefficient is indeterminate

$$\therefore\qquad \frac{dy}{dx}=-\left(\frac{du}{dx}\right)\div\left(\frac{du}{dy}\right)=\frac{0}{0}.\ \ldots\ldots\ldots\ldots\ (c)$$

Evaluate this vanishing fraction

$$\therefore\ \frac{dy}{dx}=-\left\{\left(\frac{d^2u}{dxdy}\right)\frac{dy}{dx}+\left(\frac{d^2u}{dx^2}\right)\right\}\div\left\{\left(\frac{d^2u}{dy^2}\right)\frac{dy}{dx}+\left(\frac{d^2u}{dxdy}\right)\right\}\ldots(d)$$

(Todhunter's Differential Calculus, art. 190).

Eliminate $\frac{dy}{dx}$ by eq. (c) and clear of fractions and we again obtain $H=0$.

$\therefore$ the Hessian also passes through the double points of the curve $u = 0$.

A triple point is the superposition of three double points and any multiple point of the order k is the superposition of k double points.

3° Let us move the origin of $u = 0$ to a double point and make the two tangents at the point the coordinate axes. Then $u = 0$ has no constant term or terms of the first degree, and no terms of the second degree except one containing xy, (Eddy's Anal. Geom. arts. 458, 453, 424), and may be written in the form $axy + t_{n>2} = 0, \ldots \ldots \ldots \ldots \ldots (e)$

in which $t_{n>2}$ signifies "terms of higher degree than the second."

Now compute the Hessian of (e)

$$\left(\frac{du}{dx}\right) = ay + t_{n>1}, \quad \left(\frac{du}{dy}\right) = ax + t_{n>1}, \quad \left(\frac{d^2u}{dx^2}\right) = t_{n>0},$$

$$\left(\frac{d^2u}{dxdy}\right) = a + t_{n>0}, \quad \left(\frac{d^2u}{dy^2}\right) = t_{n>0}.$$

$\therefore$ substituting, $H = 2a^3xy + t_{n>2} = 0$, which also has the axes both tangent to it for the same reason as eq. (e). Hence at a multiple point of $u = 0$ the tangent of each branch is also tangent to a branch of the Hessian, and a multiple point of the order k in the curve $u = 0$ is a multiple point of the same order in $H = 0$.

4°. Let us move the origin of $u = 0$ to a cusp and make the axis of x the double tangent at the cusp. Then there is no constant term in the equation, or terms of the first degree, and no term of the second degree except that containing y^2; as appears from considerations like those used in obtaining eq. (e). It may be written $ay^2 + t_{n>2} = 0. \ldots \ldots \ldots \ldots (f)$

Compute the Hessian of (f)

$$\left(\frac{du}{dx}\right) = t_{n>1}, \quad \left(\frac{du}{dy}\right) = 2ay + t_{n>1}, \quad \left(\frac{d^2u}{dx^2}\right) = t_{n>0},$$

$$\left(\frac{d^2u}{dxdy}\right) = t_{n>0}, \quad \left(\frac{d^2u}{dy^2}\right) = 2a + t_{n>0}.$$

$\therefore H = t_{n>2} = 0$, which has in it no terms of the zero, first or second degrees, therefore it has three branches through the cusp. (Eddy's Anal. Geometry, art. 468). Hence at a cusp of $u = 0$ there is a triple point of $H = 0$, and two branches of the Hessian have a common tangent with the cusp.

5°. Similarly move the origin to a point of inflection and $u = 0$ becomes, when the axis of x is tangent at the point,

$$ay + bxy + cy^2 + t_{n>2} = 0, \quad \ldots\ldots\ldots\ldots (g)$$

which does not contain x^2. (Anal. Geom. art. 425). Compute the Hes. of (g),

$$\left(\frac{du}{dx}\right) = by + t_{n>1}, \quad \left(\frac{du}{dy}\right) = a + bx + 2cy + t_{n>1}, \quad \left(\frac{d^2u}{dx^2}\right) = t_{n>0},$$

$$\left(\frac{d^2u}{dxdy}\right) = 2c + t_{n>0}, \qquad \left(\frac{d^2u}{dy^2}\right) = b + t_{n>0}.$$

$\therefore H = t_{n>0} = 0$; and since there is no constant term the Hessian passes through the point of inflection as before stated.

This method may be applied in a similar manner to the singularities of higher orders which, as has been shown by Cayley, are merely the superposition or coincidence of several of these simple singularities.

BIPOLAR EQUATIONS – CARTESIAN OVALS.

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Every system of co-ordinates is specially adapted to the expression of some particular properties of those curves which admit of simple equations in that system. The object of this paper is to discuss the Bipolar system of point-coordinates and particularly the Cartesian Ovals, (the loci of Bipolar equations of the first degree).

Let the fixed points A_1 and A_2 be taken as poles, and denote by a the distance between the poles; then ρ_1 and ρ_2, denoting the distances of any point from the poles, are the bipolar coordinates of that point. We observe that any two points symmetrically situated with respect to the line A_1A_2 have the same pair of coordinates, hence the locus of every equation in bipolar coordinates is symmetrical with respect to the axis A_1A_2.

The equations of transformation to rectangular and to polar coordinates, taking A_1 for origin or pole and A_1A_2 for axis of x or initial line, are

$$\left.\begin{aligned}\rho_1^2 &= x^2 + y^2 = \rho^2 \\ \rho_2^2 &= x^2 + y^2 - 2ax + a^2 = \rho^2 - 2a\rho\cos\theta + a^2\end{aligned}\right\} \ldots\ldots (1)$$

The values of ρ_1 and ρ_2 in terms of x and y being square roots, we observe that the rational rectangular equation obtained by transformation from any bipolar equation will always include the several loci which result from giving to ρ_1 and ρ_2 the ambiguous sign. Thus $\pm\rho_1 \pm \rho_2 = na \ldots (2)$ will be included in a single rectangular equation. We shall therefore regard the results of different selections of the signs as the equations of

different branches of the same curve. These branches are not all possible at once, thus in (2) if $n>1$, $\rho_1 + \rho_2 = na$ is the only possible branch, (since for every point we must have $\rho_1 \sim \rho_2 < a$) and represents an ellipse; while if $n<1$, this branch is impossible and we have $\rho_1 - \rho_2 = \pm\, na$ representing the two branches of an hyperbola. Thus (2) is the general equation of a conic with foci at the poles.

Bipolar equations may sometimes be rendered homogeneous by the introduction of a third coordinate, ρ_3, representing the distance of the point from a third pole A_3, taken on the axis A_1A_2; thus forming a redundant system of coordinates analogous to the trilinear system. To deduce the relation which must always exist between ρ_1, ρ_2, and ρ_3, denote in Fig. 1 the distance between A_1 and A_2 by a_3, between A_2 and A_3 by a_1, between A_3 and A_1 by a_2; for the sake of symmetry so taking the signs of these quantities that

$$a_1 + a_2 + a_3 = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

(In the figure a_2 is regarded as negative.) Denote the projection of ρ_3 upon the axis by x; then

$$\rho_1^2 - \rho_3^2 = (a_2 + x)^2 - x_2$$
$$= a_2^2 + 2a_2x$$
$$\rho^2_2 - \rho_3^2 = (a_1 - x)^2 - x_2$$
$$= a_1^2 - 2a_1x.$$

Eliminating x, $\quad a_1\rho_1^2 - a_1\rho_3^2 + a_2\,\rho_2^2 - a_2\rho_3^2 = a_1a_2^2 + a_2a_1^2$,

or since $a_1 + a_2 = -a_3$

$$a_1\rho_1^2 + a_2\rho_2^2 + a_3\rho_3^2 + a_1a_2a_3 = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

By means of this relation we may eliminate the absolute term from a bipolar equation. We may use it also to eliminate one of the variables, for instance ρ_2, from a bipolar equation thus obtaining the bipolar equation of the same locus referred to the poles A_1 and A_3.

The general equation of the first degree may be written in the form

$$\pm\, m\rho_1 \pm l\rho_2 \pm na = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

in which we regard each of the symbols as denoting a positive quantity. The locus of (5) consists of two branches, (except when the equation becomes that of an ellipse,) and two only. For in the first place it is obvious that the terms can not all have the same sign, we may therefore write two of the terms with the positive and one with the negative sign; now since ρ_1, ρ_2 and a form the sides of a triangle, of which one side can not exceed the sum of the other two, it is impossible that the negative term, which must in absolute value equal the sum of the other two, should have the least of the three coefficients, m, l, and n. We therefore obtain the equations of the real branches by giving to *one or the other of the two greater coefficients the negativ sign.*

Infinite values of ρ_1 and ρ_2, it is evident, can never satisfy the equation unless the coefficients m and l are equal and have opposite signs, in which case the curve becomes an hyperbola. In all other cases therefore the locus consists of two closed branches which are called Cartesian Ovals.

One pole at least is within both ovals. For since either the sum or the difference of $m\rho_1$ and $l\rho_2$ is constant we have for any two points P and P' of the same branch, $\qquad m(\rho_1 \sim \rho'_1) = l(\rho_2 \sim \rho'_2)$.
Now let P and P' be the points in which the oval cuts the axis. If the pole A_1 is outside of the oval, the difference $\rho_1 - \rho'_1$ becomes the axis of the oval, if it is within it becomes the difference between two segments of the axis of the oval, and will be the less the nearer the pole is to the middle point of this axis. Now if $m > l$, the difference $\rho_1 \sim \rho'_1$ will be less than $\rho_2 \sim \rho'_2$; hence A_1 will be within the oval, and will be nearer than A_2 is to the middle point, in case the latter also is within the oval. That is, *corresponding to the greater of the coefficients m and l, we have a pole which is within each oval.*

Supposing $m > l$, if $n > l$ the equations of the branches will be

$$\left.\begin{aligned} m\rho_1 + l\rho_2 - na = 0, \\ -m\rho_1 + l\rho_2 + na = 0. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (6)$$

In the first $m\rho_1$ is always less than the constant na, in the second it is always greater than na; therefore since A_1 from which ρ_1 is measured is within both branches, the first branch lies entirely within the other. If however $m = n$, the branches will meet in the single point A_2 whose coordinates $(a, 0)$ satisfy both equations.

If $m > l > n$, the equations will be

$$\left.\begin{aligned} m\rho_1 - l\rho_2 + na = 0, \\ -m\rho_1 + l\rho_2 + na = 0. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (7)$$

In this case *the pole A_2 will be outside of each oval.* For, at any point between A_1 and A_2 we have $\rho_1 + \rho_2 = a$, combining this with equations (7) we have

$$\rho_1 = \frac{l \mp n}{m + l}a. \quad \ldots\ldots\ldots\ldots\ldots \quad (8)$$

Each of these values is positive and less than a; therefore the branches cut A_1A_2 between the poles, and A_1 being within, A_2 must be outside of each oval. The greater value of ρ_1 in (8) is derived from the second equation in (7), therefore that equation belongs to the outer branch. In both the cases (7) and (8), *the outer branch is that in which the greater coefficient has the negative sign.*

We will now show that the Cartesian

$$m\rho_1 \pm l\rho_2 \pm na_2 = 0 \quad \ldots\ldots\ldots\ldots \quad (9)$$

admits of a homogeneous tripolar equation of the first degree. Transposing and squaring (9) $m^2\rho_1^2 \pm 2ml\rho_1\rho_2 + l^2\rho_2^2 = n^2a_3^2$.
This equation is rendered homogeneous by multiplying by

$$1 = -\frac{\rho_1^2}{a_2a_3} - \frac{\rho_2^2}{a_3a_1} - \frac{\rho_3^2}{a_1a_2}$$

derived from (4), which gives

$$\left(m^2 + n^2\frac{a_3}{a_2}\right)\rho_1^2 \pm 2ml\rho_1\rho_2 + \left(l^2 + n^2\frac{a_3}{a_1}\right)\rho_2^2 = -n^2\frac{a_3^2}{a_1a_2}\rho_3^2 \dots (10)$$

This equation will reduce to one of the first degree when the first member is a square, that is, when

$$\left(m^2 + n^2\frac{a_3}{a_2}\right)\left(l^2 + n^2\frac{a_3}{a_1}\right) = m^2l^2,$$

or $$l^2a_1 + m^2a_2 + n^2a_3 = 0. \dots\dots\dots\dots (11)$$

The values of a_1 and a_2 must also satisfy (3), $a_1 + a_2 + a_3 = 0$,

hence we derive $$a_1 = \frac{n^2 - m^2}{m^2 - l^2}a_3 \quad\text{and}\quad a_2 = \frac{l^2 - n^2}{m^2 - l^2}a_3. \dots\dots (12)$$

To reduce equation (10) most symmetrically, multiply by a_1a_2,

$$a_1(m^2a_2 + n^2a_3) \pm 2mla_1a_2\rho_1\rho_2 + a_2(l^2a_1 + n^2a_3)\rho_2^2 = -n^2a_3^2\rho_3^2,$$

substituting from (11), $-l^2a_1^2\rho_1^2 \pm 2mla_1a_2\rho_1\rho_2 - m^2a_2^2\rho_2^2 = -n^2a_3^2\rho_3^2$.

Hence $$la_1\rho_1 \mp ma_2\rho_2 \pm na_3\rho_3 = 0. \dots\dots\dots (13)$$

The three poles in this equation constitute three foci of the Cartesian, with respect to any pair of which the curve has similar properties. For, eliminating ρ_2 between (13) and (9) [observing that in the first ambiguous signs of these equations, the upper signs correspond], we have

$$(l^2a_1 + m^2a_2)\rho_1 \pm mna_2a_3 \pm nla_3\rho_3 = 0,$$

or by (11) $$-n^2a_3\rho_1 \pm mna_2a_3 \pm nla_3\rho_3 = 0$$

or $$\pm n\rho_1 \pm ma_2 \pm l\rho_3 = 0.$$

The three bipolar and the tripolar equation are therefore as follows—

$$\left.\begin{aligned} \pm m\rho_1 \pm l\rho_2 \pm na_3 &= 0 \\ \pm n\rho_1 \pm ma_2 \pm l\rho_3 &= 0 \\ \pm la_1 \pm n\rho_2 \pm m\rho_3 &= 0 \\ \pm la_1\rho_1 \pm ma_2\rho_2 \pm na_3\rho_3 &= 0 \end{aligned}\right\} \dots\dots\dots (14)$$

Since the three bipolar equations have the same coefficients their order only being changed, we see that the distinction between the cases (6) and (7) is not a distinction between varieties of the curve, but is due solely to the relative position of the foci which are taken as poles. Let the order of magnitude of the coefficients be $n > m > l$; the first equation shows that A_1 is within the ovals and nearer to the middle than A_2, while the third equation, in which the coefficient of the absolute term is the least, (as in eq. (7),)

shows that A_2 is within and A_3 outside of the ovals. We may call A_1, in this case, the first focus, A_2 the second, and A_3 the third focus.

When $m=l$, the Cartesian $m\rho_1 \pm l\rho_2 \pm na_3 = 0$ becomes a conic, and (12) shows that the pole A_3 is at infinity. The properties expressed in the second and third bipolar equations of (14) then become the property of a focus and directrix. This may be proved as follows—Writing the equation in the form
$$ee'\rho_1 \pm e\rho_2 \pm a_3 = 0$$
the second equation of (14) becomes
$$\rho_1 \pm ee'a \pm e\rho_3 = 0 \text{ or } \qquad \rho_1 = \pm e(e'a_2 \pm \rho_3).$$
When $e>1$ and e' near to 1, the two branches take the forms
$$\rho_1 = \pm e(e'a_2 - \rho_3). \quad \ldots \ldots \ldots \ldots \quad (15)$$
If $e<1$ and e' near to 1, the inner branch will take one of the forms in (15). Now $e'a_2 - \rho_3$ expresses the distance of a point from the circumference of a circle whose centre is A_3, and (15) expresses that the distance of any point of the curve from A_1 is in the fixed ratio e to its distance from this circumference. When $e'=1$, this circumference becomes a straight line and e is the excentricity of the conic, which consists of two branches or one according as $e>1$ or $e<1$.

Transforming to polar coordinates by (1) after squaring the equation
$$m\rho_1 \pm na_3 = \mp l\rho_2$$
we obtain
$$\rho^2 + \frac{2l^2a_3}{m^2-l^2}\rho\cos\theta \pm \frac{2mna_3}{m^2-l^2}\rho + \frac{n^2-l^2}{m^2-l^2}a_3{}^2 = 0, \quad \ldots \quad (16)$$
the polar equation of the Cartesian referred to one of its foci and the axis.

If $n=l$, (16) reduces to $\rho=0$ and
$$\rho + \frac{2l^2a_3}{m^2-l^2}\cos\theta \pm \frac{2mla_3}{m^2-l^2} = 0. \quad \ldots \ldots \ldots \quad (17)$$
This last is the equation of the Limaçon of Pascal or curve generated by increasing and diminishing the radius vector of a circle (pole in the circumference) by the same constant. The bipolar equation
$$m\rho_1 \pm l\rho_2 \pm la_3 = 0 \quad \ldots \ldots \ldots \ldots \quad (18)$$
therefore represents the limaçon accompanied by its node. Making $n=l$ in (12) we see that A_3 coincides with the node at A_1. If $m>l$, the first and second foci coincide at an acnode or conjugate point, A_1 $(0, a_3)$ being the only point which satisfies the equation $m\rho_1 \pm l(\rho_2 - a_3) = 0$ while $-m\rho_1 + l\rho_2 + la_3 = 0$ represents the continuous curve. If $m<l$, A_1 satisfies both branches, the second and third foci coinciding at a crunode or double point.

If $n=0$, the first bipolar equation becomes $m\rho_1 - l\rho_2 = 0 \quad \ldots (19)$ and the distances of the third pole from A_1 and A_2 are, making $n=0$ in (12),

$$a_2 = \frac{l^2}{m^2 - l^2} a_3 \qquad \text{and } a_1 = \frac{-m^2}{m^2 - l^2} a_3 \, . \; . \; . \; . \; . \; (20)$$

Both the second and third bipolar equations of (14) now reduce to

$$\rho_3 = \pm \frac{ml}{m^2 - l^2} a_3 .$$

Eq. (19) therefore represents a circle the third pole now coinciding with the centre, and the radius being a mean proportional between the distances (20) of the poles from the centre.

This point we shall designate by C; a_1 and a_2 being of opposite signs, C cannot be between the poles A_1 and A_2; for in $a_1 + a_2 + a_3 = 0$ the distance between the extreme points differs in sign from each of the other distances. We may therefore denote the absolute values of CA_1 and CA_2 by l^2c and m^2c, $A_1A_2 = (m^2 - l^2)c$, and from (12) we see that for any Cartesian the distances a_1 a_2 and a_3 are $n^2c - m^2c$, $m^2c - l^2c$ and $l^2c - n^2c$, thus it appears that C is a definite point for a given Cartesian being on the same side of all three poles and at distances from them proportional to l^2, m^2 and n^2; the coefficients being associated with the poles as they are in the tripolar equation of (14). The first focus is therefore the nearest to, and the third farthest from C.

C is the centre of the primitive circle of the limaçon (17), for using the notation just introduced the equation of that circle is

$$\rho + 2l^2c \cos\theta = 0$$

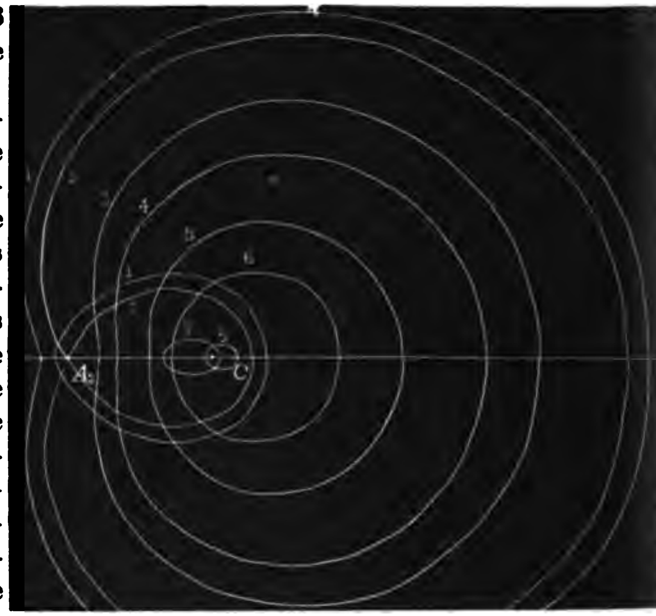
the distance of its centre from A_1 is therefore $-l^2c$, but this is the value of the distance CA_1 and the negative sign shows that it is measured in the direction toward C since we have taken the direction from C to either of the poles as positive.

The Cartesians $\qquad 2\rho_1 \pm \rho_2 \pm na = 0$

may be constructed by a method analogous to that for constructing an ellipse or hyperbola with a thread (The thread passing from the pencil, to which it is attached by means of a hole drilled in the lead, around the needle at A_1 and back to the pencil). The curves in the accompanying diagram have been constructed in this manner for various values of n. The poles A_1 and A_2 and the values of l and m, being common to the whole series the point C is also common. In No. 1 $n > 2$ the third pole is on the left of A_2 and is the third focus. When $n = 2$, the third pole coincides with A_2 and we have No. 2, the crunodal limaçon. (This curve does not generally pass through C). In No. 3 $2 > n > 1$, the third pole has passed to the right of A_2 and is the second focus. When $n = 1$, it coincides with A_1 and we have No. 4 the acnodal limaçon, the inner branch having contracted to the conjugate point A_1. In No. 5, $n < 1$, the third pole has passed to the right of

A_1 and is the first focus. Finally when $n=0$, both branches coincide with the circle No. 6 and the third pole coincides with C.

It will be observed that the conic and the limaçon fulfil the same condition as to the relatve values of the distances from C to the three poles, but in the case of the conic the absolute values of these distances become infinite. Accordingly we see in the inner branches of 3 and 5 an approximation to the elliptic form, and in 1 we have an approximation to the form of an hyperbola in the neighborhood of the second and third foci.



If s denote the arc of the Cartesian and ϕ_1 denote the angle the curve makes with ρ_1, evidently $\cos\phi_1 = \frac{d\rho_1}{ds}$ &c.

Differentiating the bipolar equation $m d\rho \pm l d\rho_2 = 0$, whence

$$m\cos\phi \pm l\cos\phi_2 = 0.$$

Thus the direction of the curve makes angles with the focal distances whose cosines are inversely as the coefficients in the bipolar equation or directly as the coefficients in the tripolar equation. When applied to the focus at infinity in the case of the conic we have the property that the cosine of the angle between the tangent and focal radius vector equals the product of the eccentricity by the cosine of the angle between the tangent and axis.

In the polar equation (16) a single value of θ corresponds to four values of ρ, denoting these by ρ' and ρ'' corresponding to the upper, and ρ''' and ρ'''' to the lower sign we have

$$\rho'\rho'' = \rho'''\rho'''' = \frac{n^2 - l^2}{m^2 - l^2} a_2^2 \quad \ldots\ldots\ldots\ldots (21)$$

$$\rho' + \rho'' = \frac{-2a}{m^2 - l^2}\left(l^2\cos\theta + mn\right),\ \rho''' + \rho'''' = \frac{-2a}{m^2 - l^2}\left(l^2\cos\theta - mn\right) \ldots (22)$$

Now since $\rho'\rho''=\rho'''\rho''''$ it is evident that for a pole within the ovals ρ' and ρ'' must belong to different branches while for a pole outside of the ovals they must belong to the same branch. Since $\rho'\rho''$ is constant the inverse of the curve described by ρ' is similar to that described by ρ''. Thus the inverse of a Cartesian with respect to either focus is a similar curve, the inner inverting into an outer branch if the focus is the first or second, but into an inner branch if it is the third focus. Thus the inverse of the acnodal limaçon with respect to the double focus is an ellipse similar to that which vanishes at that point, but its inverse with respect to the single focus is similar to itself. The inverse of the crunodal limaçon with respect to the double point which is at once on each branch consists of the two branches of an hyperbola with asymptotes parallel to the tangents at the double point. From (21) we also see that for the third focus the product of a secant and its external segment is constant, and the four tangents to the curve from this point are equal.

From (22) we have $\rho'+\rho''-\rho'''-\rho''''=$ a constant. . . . . . (23) Now when A_1 is the first focus l is the least of the three coefficients and by (21) ρ' and ρ'' have the same sign, while the pole being within the ovals ρ''' and ρ'''' are of the opposite sign. Let ρ''' belong to the same branch as ρ', then $\rho'-\rho'''$ and $\rho''-\rho''''$ will denote the parts of a straight line passing through A_1 intercepted by the ovals and will have the same sign. Therefore (23) expresses that the *sum* of the intercepts of the ovals upon a line passing through the *first* focus is constant. If A_1 is the second focus, l is the middle coefficient and ρ' and ρ'' have opposite signs. Therefore (23) expresses that the *difference* of the intercepts upon a line passing through the *second* focus is constant. If A_1 is the third focus $\rho'-\rho'''$ and $\rho''-\rho''''$ will be the intercepts between the branches, and will be of opposite signs since ρ' and ρ'' now belong to the same branch; therefore (21) will express that for a line passing through the *third* focus the *difference* of the intercepts *between* the two branches is constant.

The bipolar equation of the Cartesian takes a simple form also when referred to a focus and the point C as poles. To transform to A_1 and any other point of the axis, combine the equation with the general relation (4) so as to eliminate ρ_2. Thus, squaring, we have

$$m^2\rho_1^{\,2} \pm 2mna_3\rho_1 + n^2a_3^{\,2} = l^2\rho_2^{\,2};$$

multiplying by a_2 and substituting from (4),

$$(m^2a_2 + l^2a_1)\rho_1^{\,2} \pm 2mna_2a_3\rho_1 + l^2a_3\rho_3^{\,2} + n^2a_2a_3^{\,2} + l^2a_1a_2a_3 = 0.$$

This is referred to the focus A_1 and any other point A_3, giving to a_2 and a_1 the values in (20) A_3 coincides with C. Since a_2 is here the distance between the poles, we substitute from (20)

$$a_3 = \frac{m^2 - l^2}{l^2} a_2 \qquad \text{and} \qquad a_1 = -\frac{m^2}{l^2} a_2$$

giving $\rho_3{}^2 \pm \frac{2mn}{l^2} a_2 \rho_1 + \left[\frac{m^2 n^2}{l^4} - \left(\frac{m^2}{l^2} + \frac{n^2}{l^2} \right) \right] a_2{}^2 = 0 \; \ldots\ldots \; (24)$

a relation involving the square of one coordinate with the first power of the other.

If in this equation we suppose $n = 0$ or $m = 0$ we obtain the equation of a circle as before. If $n = l$, (24) becomes

$$\rho_3{}^2 \pm 2\frac{m}{l} a_2 \rho_1 - a_2{}^2 = 0, \quad \ldots\ldots\ldots\ldots (25)$$

this is therefore the bipolar equation of the limaçon referred to its node and the centre of the primitive circle. If $m < l$ we have the crunodal limaçon; if $m > l$, the lower sign gives the acnodal limaçon, the upper sign giving an equation satisfied only by A_1, the point $(0, a_2)$. If in (25) we make $m = l$ we have $\rho_3{}^2 - 2a_2\rho_1 - a_2{}^2 = 0 \; \ldots\ldots\ldots\ldots \; (26)$ the bipolar equation of the intermediate case, the Cardioid, which is therefore a Cartesian whose three foci coincide at the cusp. Since in (26) ρ_3 is the distance of a point from the centre of the primitive circle, $\rho_3{}^2 - a_2{}^2$ is the square of the tangent upon this circle; and (26) expresses that the square of the tangent from any point of the Cardioid to the primitive circle is a mean proportional to the diameter and the distance of the point from the cusp.

We may now find the polar equation of the Cartesian referred to C by transformation from (24), making, see (1), $\rho_3 = \rho$ and $\rho_1{}^2 = \rho^2 - 2a_2\rho\cos\theta + a_2{}^2$. For the sake of abridgement we put for the present

$$\frac{mn}{l^2} = p \qquad \text{and} \qquad \frac{m^2}{l^2} + \frac{n^2}{l^2} = s,$$

(22) then becomes $\rho_3{}^2 \pm 2pa_2\rho_1 + (p^2 - s)a_2{}^2 = 0.$
Transforming by the above

$$[\rho^2 + (p^2 - s)a_2{}^2]^2 = 4p^2 a_2{}^2(\rho^2 - 2a_2\rho\cos\theta + a_2{}^2)$$

expanding $\rho^4 - 2(s + p^2)a_2{}^2\rho^2 + 8p^2a_2{}^3\rho\cos\theta + [(p^2 - s)^2 - 4p^2]a_2{}^4 = 0.$
By adding and subtracting $(s+p^2)^2a_2{}^4$ we may write this equation in the form

$$[\rho^2 - (s + p^2)a_2{}^2]^2 + 8p^2a_2{}^3\rho\cos\theta - 4p^2(s + 1)a_2{}^4 = 0.$$

In rectangular coordinates

$$[x^2 + y^2 - (s + p^2)a_2{}^2]^2 + 8p^2a_2{}^3[x - \tfrac{1}{2}(s + 1)a_2] = 0.$$

In this equation $a_2 = l^2c$ the value of the distance CA_1; substituting and restoring the values of p and s,

$$[x^2 + y^2 - (l^2m^2 + m^2n^2 + n^2l^2)c^2]^2 + 8l^2m^2n^2c^3[x - \tfrac{1}{2}(m^2 + n^2 + l^2c)] = 0 \ldots (27)$$

an equation of the form $S^2 + k^3L = 0$ where $S = 0$ is the equation of a circle whose centre is at C, and $L = 0$ is the equation of a straight line perpendicular

to the axis. If we denote the distances CA_1, CA_2, CA_3, by a_1, a_2 and a_3 we may substitute in (27) $l^2c = a_1$ $m^2c = a_2$ $n^2c = a_3$ which gives

$$[x^2+y^2-(a_1a_2+a_2a_3+a_3a_1)]^2+8a_1a_2a_3[x-\tfrac{1}{2}(a_1+a_2+a_3)]=0\ldots.(28)$$

If one of the constants as $a_1 = 0$, the equation reduces to $S^2 = 0$, the equation of a pair of circles coincident with $S = 0$. The form of the equation $S^2 + k^2L = 0$ shows that the Cartesian touches the line $L = 0$ in the two points where this line cuts the circle $S = 0$. Hence $L = 0$ is the double tangent of the outer branch.

If we now transform the equation to any other rectangular axes it will take the form $S'^2 + k^2L' = 0$ in which $S' = 0$ is the transformed equation of $S = 0$ and $L' = 0$, that of $L = 0$; therefore the general equation of the Cartesian in rectangular coordinates is of the form $S^2 + k^2L = 0$, the centre of the circle S determining the point C, while the axis is perpendicular to the line $L = 0$. Suppose the equation when transformed to the origin C and the axis as axis of x, to be

$$(x^2 + y^2 - a^2)^2 + k^2(x - p) = 0, \ldots\ldots\ldots (29)$$

then comparing eq. (28) we see that a_1 a_2 and a_3 are the roots of the cubic

$$\alpha^3 - 2p\alpha^2 + a^2\alpha - \tfrac{1}{8}k^2 = 0. \ldots\ldots\ldots\ldots (30)$$

If in (29), $a<p$, the line $x = p$ actually touches the outer branch which has a reentrant portion like Nos. 1, 2 and 3 in the diagram. If $a = p$ which is the case in No. 4, (in which $a_1 = 1$, $a_2 = 4$ and $a_3 = 1$), $x = p$ has contact of the third order with the curve. If $a < p$, as in No. 5, the double tangent does not really touch the curve.

It is noticeable that with given values of a and p in (29) the Cartesian cannot be made to approach as near as we choose to the coincident circles $S^2 = 0$ when these are cut by $L = 0$, for the whole Cartesian is always on one side of $L = 0$; yet $k = 0$ reduces (29) to $S^2 = 0$. In fact in this case, since $p < a$, the cubic (30) will be found to have but one real root when $k = 0$. Nevertheless taking $a_3 = 0$, real values of a_1 and a_2 can be found. For comparing (28) and (29) we see we are no longer bound to satisfy the condition $p = \frac{1}{2}(a_1 + a_2 + a_3)$ when $k = 0$, hence we have only to fulfil the condition $a_1a_2 + a_2a_3 + a_3a_1 = 0$ which reduces to $a_1a_2 = 0$. Thus the foci A_1 and A_2 are real and subject only to the condition that a the radius shall be a mean proportional between them.

Since in (29) the expression $x^2 + y^2 - a^2$ is the constant product of the segments of a chord or secant to the circle through (x, y) and $x - p$ is the distance to the line $x = p$, eq. (29) expresses that the distance from any point of the Cartesian to the straight line $L = 0$ is proportional to the square of the product of the segments of a chord or to the fourth power of the tangent to the circle $S = 0$.

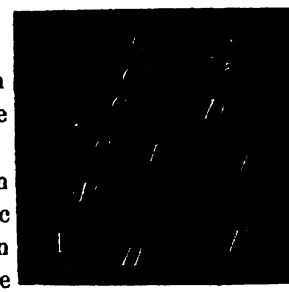
ON SUPPLYING OMISSIONS IN LAND SURVEYING.

BY P. H. PHILBRICK, PROF. OF CIVIL ENGINEERING, IOWA STATE UNIV.

In our standard works on Land Surveying we are told that: "Any two omissions in a *closed* survey whether of the length, or of the direction, or of both, of one or more of the sides bounding the area surveyed, can always be supplied by a suitable application of the principle of Latitudes and Departures."

Let us proceed to examine the truthfulness of the statement.

Case I. When the length of one side and the bearing of another are wanting.



(a). Let the deficient sides adjoin each other.

Let $ABCDEF$ (Fig. 1) represent the case; in which the *length* of BC and the *bearing* of DC are unknown.

Solution. With D as a center and the known length of DC as a radius describe an arc. This arc will in general intersect the indefinite line BC in two points (C & C') either one of which may be taken, consistently with the data, as the missing corner, thus giving the two figures $ABCDEF$ and $ABC'DEF$ either one of which may be considered as the field surveyed.

(b). Let the deficient sides be separated from each other.

Suppose the length of BC and the bearing of ED unknown.

Solution. Draw DH parallel to BC. With E as a center and the known length of ED as radius describe an arc which arc will generally intersect DH in two points D and D' from which we draw DC and $D'C''$ giving the two figures $ABCDEF$ and $ABC''D'EF$, the latter of which complies with all the data of the problem equally well with the first which by hypothesis represents the tract surveyed.

In case the length of the side whose *bearing* is unknown is equal to the length of the perpendicular upon the side whose *length* is unknown; then whether the deficient sides are separated or not the two solutions unite in one. Thus if DC is equal to the perpendicular DC''' upon BC the points C and C' coincide in C''', and the two figures $ABCDEF$ and $ABC'DEF$ become identical being represented by $ABC'''DEF$.

If more than one *known* side intervene between the deficient sides a similar construction will nevertheless suffice.

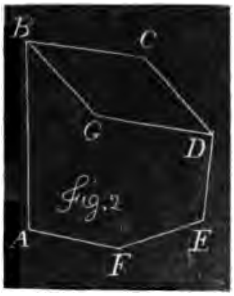
Case II. When the lengths of two sides are wanting.

(a). The deficient sides adjoin.

In Fig. 1 let the lengths of BC and DC be wanting. From B and D with the given bearings of BC and DC we must run the lines BC and DC to their intersection C, thus there is but one solution in this case.

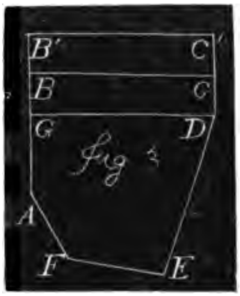
(b). The deficient sides are separated.

Let AB and DC (Fig. 2) running from A and D in known directions, be the sides whose lengths are unknown, B and C representing the undetermined corners.



From D draw DG with the length and bearing of the side which lies between AB and DC. Draw GB parallel to DC to intersect AB in B, and BC parallel to GD to intersect DC in C. $ABCDEF$ represents the tract surveyed.

If however AB and DC are parallel G is on AB or its prolongation; for if not, GB and AB, which according to the present hypothesis are parallel, would not intersect and consequently the survey would not close. Since G then, lies upon AB, and GB and AB are *parallel* they must coincide and therefore any point upon AB may be taken as their intersection from which a line drawn parallel to GD to intersect DC will close the survey. Thus there are an infinite number of solutions in this case as is illustrated in fig. 3. If more than one side lie between the deficient sides a similar construction will still suffice.



Case III. When the bearing of two sides is wanting.

(a). Let the deficient sides adjoin each other. (fig. 4)

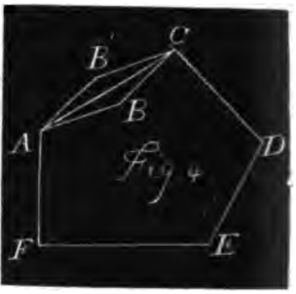
Let C, D, E, F and A represent the known corners. The distances of the corner B from A and C being known but not its bearings.

Solution. With A and C as centers and radii equal to AB and CB respectively describe arcs. These arcs will always intersect in two points B and B' giving the two figures $ABCDEF$ and $AB'CDEF$, either of which answers to all the conditions of the problem and therefore may be taken to represent the tract surveyed.

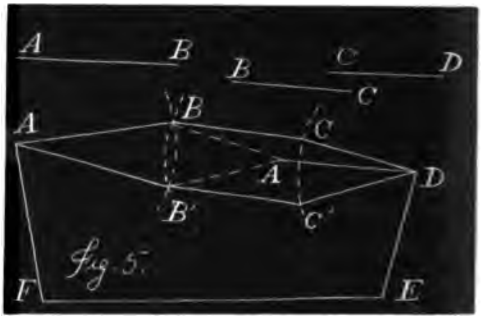
The arcs described with the centers A and C must intersect since the survey must close, and they must intersect in two points for otherwise AB and BC would represent the same line.

(*b*). Let the deficient sides be separated from each other.

Let A, F, E and D (Fig. 5) represent the fixed corners; and let AB and CD represent the length of the sides of which the bearings are wanting, and BC represent the length and bearing of the intermediate side.



Solution. With A and D as centers and radii equal respectively to AB and DC, describe arcs BB' and CC'. Draw DH equal and parallel to CB and with H as a center and radius equal also to DC describe an arc BB' which will in general intersect the former arc BB' in two points B and B'. From B and B' draw BC and $B'C'$ each parallel to DH and intersecting the arc CC' in the points C and C'. Then either $ABCDEF$ or $AB'C'DEF$ will represent the survey.



If B and B' coincide there is but one solution. If the different sides are separated by more than one side a similar construction to the above would suffice.

Omitting therefore the case in which the lengths *only* of two adjoining sides are wanting, since the fact that there can be but one position for the missing corner is axiomatic; and, for the same reason, omitting the case in which the deficiencies pertain to one side; we find three cases in which there are usually two solutions though there may be but one; one case in which there is usually but one solution though there may be an infinite number of solutions; and another case in which there are always two solutions.

Thus we see that the proposition quoted at the head of this article is missleading–in fact the truth is not in it at all–and *that* surveyor therefore who pays any regard to it will be led astray and his work vitiated.

[It freequently occurs that a formula which represents the complete solution of a question cannot be definitely interpreted from the *data* used in the formula. The additional information required is however, generally, readily obtained. The Surveyor should always, when practicable, close his survey by actual measurment, and the "principle of Latitudes and Departures" used in his calculation will then enable him to discover and eliminate his principal errors. If however it is impracticable to obtain the exact bearing of one or two of his lines, their *approximate* bearing should be noted, which will be sufficient generally to obviate the ambiguity discussed above.—Ed.]

SOLUTION OF PROBLEM 67.

BY G. W. HILL.

Assume the station as the origin of coordinates, the axis of x being directed toward the centre of the base, and that of z vertical. Let a be the radius of the base, c the altitude. The equation of the mountain's surface is then

$$a^2(c-z)^2 = c^2[(a-x)^2 + y^2].$$

The equation in terms of polar coordinates is obtained by putting

$$x = r\cos\theta\cos\omega, \qquad y = r\cos\theta\sin\omega, \qquad z = r\sin\theta,$$

and thus is
$$r = 2ac\frac{c\cos\theta\cos\omega - a\sin\theta}{c^2\cos^2\theta - a^2\sin^2\theta}.$$

The element of volume of the mountain may be regarded as a rectangular solid whose sides are dr, $rd\theta$, $r\cos\theta\, d\omega$, and ρ being its density the element of mass is $\rho r^2\cos\theta\, dr\, d\theta\, d\omega$. Its attraction on the unit of mass at the station is $\rho\cos\theta\, dr\, d\theta\, d\omega$. From the symmetry of the cone it is plain that the component of the mountain's attraction in the direction of the axis of y is zero; and the vertical component which diminishes the intensity of gravity at the station may be neglected. The component in the direction of the axis of x is

$$X = \rho\int\!\!\int\!\!\int \cos^2\theta\cos\omega\, dr\, d\theta\, d\omega.$$

Integrating with respect to r, the limits are $r=0$, and r=the value given by the equation of the surface. Thus

$$X = 2ac\rho\int\!\!\int\frac{c\cos\theta\cos\omega - a\sin\theta}{c^2\cos^2\theta - a^2\sin^2\theta}\cos^2\theta\cos\omega\, d\theta\, d\omega.$$

Next we integrate with respect to ω. As r must be always positive the limiting values of ω are the two roots of the equation $c\cos\omega = a\tan\theta$. Hence

$$X = 2ac\rho\int\left[\frac{c\cos^3\theta\cos^{-1}[(a\div c)\tan\theta]}{c^2\cos^2\theta - a^2\sin^2\theta} - \frac{\sin\theta\cos\theta}{\sqrt{(c^2\cos^2\theta - a^2\sin^2\theta)}}\right]d\theta.$$

The limits of integration are now from $\theta = 0$ to $\theta =$ the value given by the equation $a\tan\theta = c$. The second term within the brackets is integrable, and between the limits is $-\,a\div(a^2+c^2)$. To simplify the first term revert to the variable ω, that is put $a\tan\theta = c\cos\omega$. Then

$$X = 2c\rho\left[\int_0^{\frac{\pi}{2}}\frac{\omega d\omega}{\sin\omega[1 + (c^2\div a^2)\cos^2\omega]^{\frac{3}{2}}} - \frac{a^2}{a^2+c^2}\right].$$

The expression within the brackets is a function of $\frac{c}{a}$, calling it $F\left(\frac{c}{a}\right)$ we have

$$X = 2F\left(\frac{c}{a}\right)c\rho.$$

Now ρ' being the mean density and R the radius of the earth, the force of gravity g is

$$g = \frac{4\pi}{3}\rho' R,$$

and δ the deflection of the plumb-line is given by the equation

$$\tan\delta = \frac{X}{g} = \frac{3F(c \div a)}{2\pi}\,\frac{\rho}{\rho'}\,\frac{c}{R}.$$

The definite integral $\int_0^{\frac{\pi}{2}} \frac{\omega d\omega}{\sin\omega[1 + (c^2 \div a^2)\cos^2\omega]^{\frac{3}{2}}}$,

it appears, must be computed by mechanical quadratures. Here is an opportunity to illustrate the use of the formula given at p. 9, Vol. II of the ANALYST.

Divide the interval between the limits 0 and $\frac{1}{2}\pi$ into 9 equal parts; then $h = 10° = 0.1745241$. Compute the value of the function to be integrated multiplied by h for the middle of each of these parts, that is for $\omega = 5°, 15°, 25°, \ldots\ 115°$. The three values beyond 90° are for the sake of the differences. We get

ω.	Δ^0.	ω.	Δ^0.
5°	0.1400956	65°	0.2094292
15	0.1432880	75	0.2327701
25	0.1497300	85	0.2594408
35	0.1595134	95	0.2899632
45	0.1727216	105	0.3258781
55	0.1893800	115	0.3705285

As the function integrated remains the same when the sign of ω is changed, all the odd orders of differences vanish for the argument $\omega = 0$. Then making $\Delta^{-1} = 0$ for the argument $\omega = 0$, by summing and differencing we get for the argument $\omega = 90°$

$$\Delta^{-1} = 1.6563687, \qquad \Delta^1 = +\,0.0305224,$$
$$\Delta^3 = +0.0015408, \qquad \Delta^5 = +\,0.0007833.$$

Thus the value of the definite integral by the formula is

$$= 1.6563687 + \frac{1}{24}\left(0.0305224\right) - \frac{17}{5760}\left(0.0015408\right) + \frac{367}{967680}\left(0.0007833\right)$$

$= 1.6576363.$

Consequently $F(0.4) = 0.7955673$, and the demanded deflection

$$\delta = 19''.21174.$$

SOLUTIONS OF PROBLEMS IN NO 3.

Solutions of problems in No. 3 have been received as follows:

From P. E. Chase, 69, 70, 72 & 73; G. M. Day, 70, 71, 72 & 74; Th. L. De Land, 72; Cadet E. S. Farrow, 69, 70, 71, 72, 73 & 74; H. Gunder, 69, 70, 71, 72 & 74; Wm. Hoover, 69, 71 & 72; Phil. Hoglan, 72; Prof. W. W. Johnson, 74; Christine Ladd, 73; F. P. Matz, 72; E. P. Norton, 69; O. D. Oathout, 70, 72 & 73; Judge Josiah Scott, 73; Prof. J. Scheffer, 69, 70, 71, 72, 73 & 74; E. B. Seitz, 69, 70, 71, 72, 73 & 74.

69.—"Given $x^2 + y^2 = 793$, . . . (1) $\sqrt[3]{(xy^2)} + \sqrt[3]{(x^2y)} = 30$, . . . (2) to find x and y by quadratics."

SOLUTION BY PROF. P. E. CHASE, PHILAD., PA.

$$x^2 \pm 2xy + y^2 = 793 \pm 2xy = \square,$$
$$\sqrt[3]{(xy)}(\sqrt[3]{x} + \sqrt[3]{y}) = 30 = 6(3+2), \quad xy = 216, \quad x = 27, \quad y = 8.$$

SOLUTION BY E. P. NORTON, ALLEN, MICHIGAN.

Cube (2), $xy^2 + 3xy[\sqrt[3]{(xy^2)} + \sqrt[3]{(x^2y)}] + x^2y = 27000$. . (3). Substituting in (3), the value of $\sqrt[3]{(xy^2)} + \sqrt[3]{(x^2y)}$, and we have $xy(x + y + 90) = 27000$. . . . (4). Put $x + y = s$ and $xy = p$, then by making the proper substitutions in (2) and (4), we have $s^2 - 2p = 793$, (5), and $p(s + 90) = 27000$, (6). Eliminate p from (5) and (6) and $s^3 + 90s^2 - 793s = 125370$. (7) Let $s = z - 30$, then by substitution, $z^3 - 3493z = 47580$, (8). Multiply (8) by z, complete the square by adding $4225z^2 + (366)^2$ to each member of the equation, reduce, and we have $z = 65$, $s = 35$ and $p = 216$. Whence we readily find $x = 27$ and $y = 8$.

70.—"Given $y^2 + z^2 = 2500$,... (1), $x^2 = 1600[130 - (x + t)]^2$, . . . (2) $xy = t[130 - (x + t)]$, . . (3), $t^2 = y^2 + (z - 40)^2$, . . . (4), to find x, y and z."

SOLUTION BY GEO. M. DAY, LOCKPORT, N. Y.

Extract square root of (2) and divide by (3), this gives $t = 40y$. . . . . (5) Eliminate t and y from (4) by means of (1) and (5) and we have a quadratic from which we find $z = 49.99$. The other quantities are readily found by substitution.

[Subsequent to the publication of this question Mr. Matz wrote to us that he had *intended* to state eqn. (2) thus; $x^2 = 1600 + [130 - (x + t)]^2$, and that he had written to several of our contributors correcting the error.

Accordingly we have received several solutions of the question as modified by Mr. Matz by substituting in eqn. (2) after 1600 the sign $+$ instead of $\times$. We subjoin a solution, of the question thus modified, by E. B. Seitz].

Solution. Multiply (2) by t^2, square (3), and we have, by subtraction, $x^2(t^2-y^2)=1600t^2$... (5). Multiplying (4) by x^2, subtracting it from (5), and extracting the square root, we have $x(z-40)=\pm 40t$. . . . . . . (6)
Adding (1) and (4), we have $t^2=4100-80z$. . . . (7)

From (7), (6) and (2) we have, respectively,

$$z=\frac{4100-t^2}{80},\ x=\pm\frac{40t}{z-40}=\pm\frac{3200t}{900-t^2},\ x=\frac{t^2-260t+18500}{260-2t}.$$

$$\therefore \frac{t^2-260t+18500}{260-2t}=\pm\frac{3200t}{900-t^2}.$$

$\therefore \quad t^4-260t^3+11200t^2+1066000t-16650000=0;$ . . . . . (8)
or $\quad t^4-260t^3+24000t^2-598000t-16650000=0.$ . . . . . (9)

Solving (8) by Descartes' method we find $t=14.16625$, or -51.23628. $\therefore z=48.74146$, or 18.43554; $x=40t\div(z-40)=64.82325$, or 95.03838; $y=11.14763$, or -46.47723. From (9) we find $t=90$, or -15.87797. $\therefore z=-50$, or 48.09863; $x=40t\div(40-z)=40$, or 78.42304; $y=0$, or -13.65731.

71.—"Given the sides a, b, c of a spherical triangle ABC to find the radii R, r, of the circumscribed and inscribed circles."

[Several of our contributors write that this question is solved in Chauvenet's Trig., and all the solutions sent give for answers

$$\tan R=\frac{2\sin\tfrac{1}{2}a\sin\tfrac{1}{2}b\sin\tfrac{1}{2}c}{\sqrt{(\sin s\sin(s-a)\sin(s-b)\sin(s-c)}};$$

$$\tan r=\sqrt{\left(\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}\right)};$$

where $s=\tfrac{1}{2}(a+b+c)$].

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72. —"Find the maximum cylinder that can be cut from a given oblate spheroid whose semi-axes are a and b."

SOLUTION BY WM. HOOVER, BELLEFONTAINE, OHIO.

Let x be the radius and $2y$ the perpendicular of the cylinder. Then its volume is $u=2\pi x^2y=$ a max. . . . . (1)
From the ellipse $a^2y^2+b^2x^2=a^2b^2$. . . . . (2)
Substituting for x^2 in (1) its value from (2), differentiating and equating to zero we find $y=\tfrac{1}{3}b\sqrt{3}$. Substitute this value of y in (2) and we get

$$x^2=\tfrac{2}{3}a^2. \qquad \therefore u=\tfrac{4}{9}\pi a^2b\sqrt{3}.$$

73.—"Find the whole number of sets of three integers having a constant sum."

SOLUTION BY CHRISTINE LADD, CHELSEA, MASS.

Let $n =$ the constant sum. Writing down the sets which contain 1,

(a)	(b)	(c)
1	1	$n-2$
1	2	$n-3$
1	3	$n-4$
. . . . .	. . . . .	. . . . .

it is readily seen that they begin to repeat themselves when $(b) = (c)$ or $(b) + 1 = (c)$ according as n is odd or even. Put $n = 2p + q$, in which p is integral and $q < 2$; i. e. $q = 1$ if n is odd and $q = 0$ if n is even.

Then $$(b) = n - 1 - (b) = \frac{n-1}{2} \text{ when } q = 1,$$

or $$(b) = n - 1 - (b) - 1 = \frac{n-1-1}{2} \text{ when } q = 0;$$

$$(b) = p + q - 1 \text{ in both cases.}$$

But the maximum value of (b) is also the number of sets containing 1.

Take next the sets which contain 2. $2 + 1 + n - 3$ has already occurred so we begin with

(a)	(b)	(c)
2	2	$n-4$
2	3	$n-7$
2	4	$n-8$
. . . . .	. . . . .	. . . . .

which will continue until

$$(b) = n - 2 - (b) = \frac{n-2}{2} \text{ when } q = 0,$$

or $$(b) = n - 2 - (b) - 1 = \frac{n-2-1}{2} \text{ when } q = 1.$$

$(b) = p - 1$ in both cases, which is one more than the number of sets containing 2, $= p - 1 - 1$.

(a)	(b)	(c)
3	3	$n-6$
3	4	$n-7$
3	5	$n-8$
. . . . .	. . . . .	. . . . .

until $$(b) = \frac{n-3}{2}\ (q = 1), \text{ or } (b) = \frac{n-3-1}{2}\ (q = 0).$$

$(b) = p + q - 2$ in both cases, and $p + q - 2 - 2 =$ number of sets containing 3.

Collecting the results we are able to deduce the law of formation.

(1)	$p+q-1$	(2)	$p-1-1$
(3)	$p+q-2-2$	(4)	$p-2-2$
(5)	$p+q-3-4$	(6)	$p-3-5$
(7)	$p+q-4-6$	(8)	$p-4-7$
(9)	$p+q-5-8$	(d)	....

Collecting similar terms and adding,

$$\begin{array}{cc} (A) & (B) \\ p+q-1-3(0,1,2,\ldots.) & p-2-3(0,1,2,\ldots.) \end{array}$$

in which it remains to find the limit of the series $(0, 1, 2, \ldots.)$.

Put $n=3(2k+m)+l$ in which k, m, l, are integral, $l<3$, $m<2$. (d) will increase up to $2k+m$. Of these variations of sets $k+m$ will be in (A) and k in (B). Hence we shall have,

$$(A)+(B)=(k+m)(p+q-1)+k(p-2)-3[0,1,2,\ldots(k+m-1)]$$
$$-3[0,1,2,\ldots(k-1)].$$

74.—"Find the equation of the locus of the middle point of a chord to the hyperbola $x^2-y^2=2a^2$, the chord being of constant length and equal to seven times the transverse axis."

SOLUTION BY PROF. W. W. JOHNSON, ANNAPOLIS, MD.

Combining the equation of the straight line $y=mx+b$ with that of the hyperbola we find for the intersections

$$x=\frac{bm\pm\sqrt{[2a^2(1-m^2)+b^2]}}{1-m^2} \text{ and } y=\frac{b\pm m\sqrt{[2a^2(1-m^2)+b^2]}}{1-m^2};$$

hence the coordinates of the middle point are

$$x=\frac{bm}{1-m^2} \quad\text{and}\quad y=\frac{b}{1-m^2}$$

and the sum of the squares of the radical parts equals the square of half the chord equals the square of seven times the transverse semi-axis; that is

$$(1+m^2)\left[\frac{2a^2}{1-m^2}+\frac{b^2}{(1-m^2)^2}\right]=98a^2.$$

Eliminating b & c $(1+m^2)(2a^2+y^2-m^2y^2)=98a^2(1-m^2)$,

but $m=\dfrac{x}{y}$ hence $\left(1+\dfrac{x^2}{y^2}\right)\left(2a^2+y^2-x^2\right)=98a^2\left(1-\dfrac{x^2}{y^2}\right)$

or $$(y^2+x^2)(2a^2+y^2-x^2)=98a^2(y^2-x^2)$$

or $$y^4-x^4+100a^2x^2-96a^2y^2=0.$$

Note. This is the curve known as "*la courbe du diable*" which is given by its equation as an example in all the books on curve-tracing. If we

change the coefficient of a^2y^2 from 96 to 100 the equation represents the straight lines $y=\pm x$ and the circle $x^2+y^2=100$, to which system of lines the curve is therefore closely asymptotic. I conjecture that the curve was originated in this way and has not heretofore been known as a geometrical locus. Can any of your readers throw any light on the history of this curious curve and its startling title?

Answer to Prof. Hall's Query, by Prof. W. W. Johnson.

Prof. Cayley proposed the question "Find the number of regions into which infinite space is divided by n planes" in the Smith Prize Examination Feb. 3rd, 1874, and published in the Mathematical Messenger for march 1874, "Solutions and remarks" on the paper of that day. He says he intended the questson for a problem, as the result, though a known, is not a generally known one. His solution is substantially as follows: Consider the analogous problem for lines in a plane. An additional line adds to the number of regions one for every part into which it is itself divided by the other lines. Hence, 1, 2, 3, 4 &c. lines divide a plane into 2, $2+2(=4)$, $4+3(=7)$, $7+4(=11)$ &c. regions; the general term being $\frac{1}{2}(n^2+n+2)$. In like manner an additional plane adds to the number of regions in space one for every region in to which it is itself divided by the other planes. Hence 1, 2, 3, 4, &c. planes divide space into 2, $2+2(=4)$, $4+4(=8)$, $8+7(=15)$, $15+11(=26)$ &c. regions; the general term being $\frac{1}{6}(n^3+5n+6)$.

[Mr. G. W. Hill obtains the same result as answer to Prof. Hall's Query and by analogous reasoning, employing however in his investigation the Calculus of Finite Differences.

It will be observed that the question as proposed by Prof. Cayley is not identical with that proposed by Prof. Hall; as Prof. Cayley requests the number of regions into which infinite space *is* divided by n planes, whereas Prof. Hall asks, "Into how many parts *can* n planes divide space."

That the answer given is not *necessarily* the answer to Prof. Cayley's question follows from the fact that nothing in Prof. Cayley's announcement of the question precludes the possibility (theoretically at least) of some or all of the planes being parallel, in which case the answer would obviously not be correct: If drawn at random, however, the probability of such a contingency is infinitely small.—Ed.]

Answer to President Tappan's Query, by Prof. J. Scheffer.

It is. The French mathematician *Fermat* who published quite a number of theorems in regard to prime numbers, erroneously asserted that all the

numbers which have the form $a^m + 1$, in which m is a power of 2, are prime numbers. He declared himself unable to give a demonstration, but he was nevertheless convinced of the truth of his assertion. The latter is correct for all numbers not exceeding five figures; for instance, 2^2+1, 2^4+1, 2^8+1, $2^{16}+1$ ($=5$, 17, 257, 65537). But $2^{32}+1=4294967297$ is no longer a prime, for, as *Euler* has proved, it is divisible by 641.

[This Query was also answered in the affirmative by G. W. Hill, Prof. Chase, Judge Scott, E. S. Farrow, O. D. Oathout, E. B. Seitz and Henry Gunder.]

Solution of Mr. Holbrook's Problem, (p. 72) by Prof. H. T. Eddy.

Suppose the surface to be generated by the line

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}. \qquad \ldots\ldots\ldots\ldots (a)$$

Let the surface have a distance of $2r$ along the axis of x for its edge and the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \qquad z=c, \qquad \ldots\ldots\ldots\ldots (b)$$

for its base, and the axis of z for its conical axis. Then if the point (x_1, y_1, z_1) is situated on the axis of x and (x_2, y_2, z_2) upon the ellipse the problem evidently requires $x_1 : r :: x_2 : a$. $\therefore x_2=\frac{ax_1}{r}$. .... (c)

Also $\quad y_1=0, \qquad z_1=0, \qquad z_2=c,$ and from (b),

$$y_2=\frac{b}{a}\sqrt{a^2-x_2^2}. \quad \therefore \text{ from } (c),\ y_2=\frac{b}{r}\sqrt{a^2-x_1^2}.$$

$$\therefore \text{ from } (a), \qquad \frac{r(x-x_1)}{x_1(a-r)}=\frac{ry}{b\sqrt{(a^2-x_1^2)}}=\frac{z}{c}. \qquad \ldots\ldots\ldots\ldots (d)$$

$$\therefore \qquad x_1=\frac{x}{\frac{z}{c}\left(\frac{a}{r}-1\right)+1}.$$

Substitute this value of x_1 in the last of equations (d) and we get

$$\frac{y}{b}=\frac{z}{c}\left[1-\frac{x^2}{r^2\left[\frac{z}{c}\left(\frac{a}{r}-1\right)+1\right]^2}\right]^{\frac{1}{2}} \qquad \ldots\ldots\ldots (e)$$

which is the equation sought.

If in (e) $a=b=r$ then $cy=z\sqrt{(r^2-x^2)}$ which is the equation of Wallis' cono-cuneus, having a circular base and an edge equal to the diameter of the base. If in (e) $r=0$ then we have

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$$

which is the equation of an elliptic cone.

NOTE ON THE + AND — SIGNS BEFORE $\sqrt{}$, BY PROF. JOHNSON.—The question raised by Prof. Judson is one of conventional notation, viz.—shall we regard $\sqrt{a}$ as indicating sometimes the positive and sometimes the negative square root of a, or definitely the positive root. The advantage of making a distinction is unquestionable, and naturally the arithmetical root is written without sign thus $\sqrt{2} = 1.41+$. So also there are n nth roots of a, but there is an advantage in restricting the symbol $\sqrt[n]{a}$ to the arithmetical nth root, to the exclusion of the imaginary roots which have their own appropriate symbols. In like manner $\sin^{-1}x$ is by a useful convention restricted to that arc whose sine is x which is between $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$; since the expressions $\pi-\sin^{-1}x$, $2\pi-\sin^{-1}x$ &c., will then express without ambiguity the other arcs of which x is the sine.

If this convention is adopted we cannot of course say that a radical equation necessarily has a root. Thus the two equations $x+\sqrt{(ax+b)}=c$ and $x-\sqrt{(ax+b)}=c$ give rise to the same quadratic, which quadratic will have two roots real or imaginary, and it may be shown of either of these roots that it must satisfy one or other of the radical equations, but in some cases both roots belong to one of the equations and no root at all to the other.

Prof. Johnson adds—"As no one has answered my paradox published in the March No., I will mention the solution of the falacy. It consists in the tacit assumption, that under the definitions, every conic has a pole. For a conic which *has* a pole, the construction in (2) is correct. Moreover every straight line has four poles, the intersections of the system of conics whose poles lie on the line. But if we pass two conics through two *given* points, it will not generally happen that these conics admit of poles, hence the line mentioned in (5) does not generally exist, and the general conclusion in (6) is founded on a non-entity."

PROBLEMS.

75. BY D. BROWN, GRAFTON, ILL.—A stick of timber of uniform density and size from end to end, has a weight of 600 pounds suspended at one end by which it is balanced horizontally on a fulcrum 6 feet from the end where the weight is suspended. If the fulcrum be 5 feet from the same end the stick will be balanced with a weight of 800 pounds. Required the weight and length of the stick.

76. BY ISAAC H. TURRELL, CUMMINSVILLE, OHIO.—Within a given circle to draw three others tangent to it and touching each other externally, the sum of whose diameters shall be equal to that of the given circle.

77. BY O. H. MERRILL, WATERTOWN, N. Y.—Within the circumference of a given circle, radius r, describe two equal circles, radii $\frac{3}{8}r$, which shall touch the first circle internally at opposite extremities of a diameter. Describe a fourth circle D, touching the first circle internally and each of the two equal circles externally. Describe a fifth circle E, a sixth circle F, &c., touching the first internally and one of the two equal circles and the circle D, E, F, &c., respectively, externally. Find an expression for the radii of the circles E, F, G, &c.

78. BY PROF. J. SCHEFFER, GAMBIER, OHIO.—From a given point without a triangle to draw a line which bisects the area of the triangle.

79. BY HENRY GUNDER, GREENVILLE, OHIO.—Prove that the formula $2^{n-1}(2^n - 1)$ represents a perfect number when $2^n - 1$ is prime.

80. BY PROF. A. B. EVANS, LOCKPORT, N. Y.—The relation

$$|\underline{n} = 1 + A_1\frac{n}{|\underline{1}} + A_2\frac{n(n-1)}{|\underline{2}} + A_3\frac{n(n-1)(n-2)}{|\underline{3}} + \ldots + A_n\frac{|\underline{n}}{|\underline{n}}$$

is true for all positive integral values of n; show that

$$\frac{A_r}{|\underline{r}} = 1 - \frac{1}{|\underline{1}} + \frac{1}{|\underline{2}} - \frac{1}{|\underline{3}} \ldots + \frac{(-1)^r}{|\underline{r}},$$

where r is an integer less than n.

81. BY G. W. Hill, NYACK TURNPIKE, N. Y.–"Prove that, *identically*,

$$\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \ldots + \frac{1}{2n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots - \frac{1}{2n}."$$

82. BY PROF. A. HALL, NAVAL OBS., WASHINGTON, D. C.—If the parabolic orbits of two comets intersect the circular orbit of the earth in the same two points, then if t_1 and t_2 be the times in which the comets move from one point to the other, $(t_1 + t_2)^{\frac{2}{3}} + (t_1 - t_2)^{\frac{2}{3}} = \left(\frac{4}{3\pi}\right)^{\frac{2}{3}}$: a year being the unit of time.

ERRATA.

On page 88, 1st line, the exponent of the factor $(1 - f')$ should be $\frac{3}{2}$.
" 91, line 13, for 396 read $396x$.
" 95, " 6 from bottom, for "*nn*." read nn'.
" 99, " 15 from bottom, for make read mark.
" 100, " 18, for f, g, read f', g',.
" 103, " 22, for number read numerator.
" 107, " 18, for $\rho_2{}^2$ read $\rho_3{}^2$.
" 109, " 14, after first parenthesis insert $\rho_1{}^2$.
" 118, Fig. 5, for "*A*" within the figure read H.

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Correction. On p. 157, line 17, for "Put $A_r = x^r = [(x+1) - 1]^r = (x+1)_r$," read, Because $x^r = [(x+1) - 1]^r = (x+1)^r$, &c.; and on same page, line 19, for "Then" read Therefore.

THE ANALYST.

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DIVISIBILITY BY SEVEN.

BY PROF. EDWARD BROOKS, MILLERSVILLE, PENN.

The *Divisibility of Numbers*, as presented by different authors, embraces the conditions of divisibility by the numbers 2, 3, etc. up to 12, with the omission of the number 7. This omission leads us to inquire whether there is any general law for the divisibility of numbers by seven. A few of our text-books present some special truths in regard to this subject, among which are the following:

1. *A number is divisible by 7 when the unit term is one half or one ninth of the part on the left.* Thus 21, 42, 63, 126, and 91, 182, 273, etc.

2. *A number is divisible by 7 when the number expressed by the two right-hand terms is five times the part on the left, or one third of it.* Thus, 840, 525, 1995, and 3612, 903, 602, etc.

There are, however, some general laws for the divisibility by 7, which seem to have been overlooked by most writers on the theory of numbers, and which, though of no particular practical importance, are interesting in a scientific point of view. The first and least simple of these laws is as follows:

I. A number is divisible by 7 when the sum of once the first or units term, 3 times the second, 2 times the third, 6 times the fourth, 4 times the fifth, 5 times the sixth, once the seventh, 3 times the eighth, etc. is divisible by 7.

It will be seen that the series of multipliers is 1, 3, 2, 6, 4, 5. To illustrate the law, take the number 7935942, and we have for the sum of the multiples of the digits, $1\times2 + 3\times4 + 2\times9 + 6\times5 + 4\times3 + 5\times9 + 1\times7 = 126$, which is exactly divisible by 7; and if we divide the number itself by 7, we find there is also no remainder.

In this law we see that the second half of the series of multipliers 6, 4, 5, equals respectively 7 minus the first half, 1, 3, 2, hence, instead of adding the multiples of the second series, 6, 4, 5, we may subtract the respective multiples of the terms of the second numerical period by the first series 1, 3, 2, which will give rise to the following principle:

II. A number is divisible by 7 when the number arising from the sum of once the first term, 3 times the second, 2 times the third, minus the sum of the same multiples of the next three terms, plus the sum of the same multiples of the next three terms, etc., is divisible by 7.

It will be seen that the series of multipliers is 1, 3, 2, the first products being additive and the second products subtractive, etc.; the multiples of the terms of the *odd* numerical periods being additive and of the *even* periods subtractive. If we take the number 5439728, we have $1\times8 + 3\times2 + 2\times7 - 1\times9 - 3\times3 - 2\times4 + 1\times5 = 7$, which is divisible by 7; and upon trial we fiind the number is also exactly divisible by 7. This second principle may be more simply stated thus:

II_a. A number is divisible by 7 when the sum of the multiples expressed by the numbers 1, 3, 2, of the terms of the odd numerical periods, minus the sum of the same multiples of the terms of the even numerical periods, is divisible by 7.

Now, if we add exact multiples of 7 to the multiples of the terms which are united in the test of divisibility, it will not change the remainder. Thus, taking the number 5439728, if we add 7×2 to 3×2, we have 10×2 or 20, and adding 98×7 to 2×7, we have 100×7 or 700; hence we may use, in place of $1\times8 + 3\times2 + 2\times7$, the sum of $8 + 20 + 700$, or 728, the first numerical period; and in the same way it may be shown that we may use the second period, subtracting in the last, etc. Hence from principle II. we may derive the following principle:

III. A number is divisible by 7 when the sum of the odd numerical periods, minus the sum of the even numerical periods, is divisible by 7.

To illustrate this principle, take the number 5,643,378,762; we have for the sum of the odd numerical periods $762 + 643 = 1405$; for the sum of the even periods $378 + 5 = 383$; the difference is 1022, which is exactly divisible by 7; and if we divide the number itself by 7, we find that there is also no remainder.

Applyingthe same reasoning to Principle I., by which we derived Principle III. from Principle II., we obtain the following Principle:

IV. A number is divisible by 7 when the sum of the double numerical periods is divisible by 7.

Thus in 5,643,378,762, the sum 378,762 + 5,643, or 384405, is divisible by 7, and the number is also divisible by 7.

These principles may be put in the more general form of determining the remainder arising from dividing by 7. Thus the third principle may be stated as follows:

V. Any number divided by 7 *leaves the same remainder as the difference of the sums of the odd numerical periods and the even numerical periods divided by* 7. If the sum of the even periods is the greater we subtract the remainder from 7 for the true remainder, as in the similar case of the principle for 11.

The second, third, and fourth principles, as here presented, are derived from the first, though they may be demonstrated independently. The arithmetical demonstration of Prin. V. leads to a still more general law of divisibility, since the factor 1001, which will be seen to be the basis of the demonstration, is the product of 7, 11, and 13. The principle is thus seen to be true for 11 and 13 and the most general form of statement is as follows:

VI. Any number, divided by 7, 11, *or* 13, *leaves the same remainder as is obtained when the sum of the odd numerical periods minus the sum of the even numerical periods is divided by these numbers.*

From this we may immediately derive the law of exact divisibility by 7, 11, and 13, namely:

VII. Any number is divisible by 7, 11, *and* 13 *when the difference between the sums of the odd and even numerical periods is divisible by these numbers.*

Subsequent to the discovery of Prin. II., I learned that Prof. Elliott had employed the same property as early as 1846; whether it was known previously to this date has not been ascertained. I have also recently noticed that Prin. III. is given by one or two writers, but with whom it originated I am not aware. The other two laws, so far as I can learn, have not been previously published.

HISTORICAL SKETCH OF AMERICAN MATHEMATICAL PERIODICALS.

BY DAVID S. HART M. D., STONINGTON, CONN.

From the settlement of the American Colonies to the commencement of the present century, but little attention was paid to Science and especially to mathematical Science. Many of the early settlers of the Colonies, and particularly the Clergy, were among the most eminent classical scholars in the world. Many of them had carried off the palm of victory at Oxford and Cambridge in England. Latin and Greek, and in some instances Hebrew, were as familiar to them as their native tongue. Some of them had even mastered the Chaldee, Syriac and Arabic languages. Theological speculation was indulged in to a great extent. In mathematics, on the other hand,

little more than the elements of Algebra, Geometry and Astronomy were taught in the Colleges, such for instance as Harvard and Yale. The Differential and Integral Calculus, and others of the higher branches of mathematics were known only to the "chosen few." In these studies America was far behind Europe. But, at the beginning of this century, several gentlemen in New York and other cities, who had long felt the want of a periodical which should do for America what the "Ladies' Diary" had for a century done for England, resolved to form an association for that purpose. They established a periodical styled "The Mathematical Correspondent" to be published quarterly in numbers containing only one sheet. Several editors were appointed, of whom George Baron was Editor in chief. The first number was issued in New York, May 1st, 1804. Eight numbers only were published, which were bound as Vol. I. In this volume there was a supplementary number containing two essays, one of which was on the Diophantine Analysis by Robert Adrain. This was the first attempt to introduce the study of this refined analysis in America. He intended to continue the subject in the next volume; but alas! that volume never appeared.

The causes of the discontinuance of this periodical are obvious even upon a slight examination. In the first place, the Editors permit a contributor who calls himself A. Rabbit, to sneer at several works written by American authors, as Shepherd, Pike, Walsh and others. They themselves also speak in the most contemptuous manner of Col. Jared Mansfield, superintendent of the Military Academy at West Point. The writer has a copy of No. 2. stitched in a blue cover, on which is an advertisement of a Lecture delivered in New York by G. Baron, which contains (as he says) "a complete refutation of the false and spurious principles, ignorantly imposed on the public, in the 'New American Practical Navigator,' written by N. Bowditch and published by E. M. Blunt." The sub-editors endorsing the above say, "We agree with the author that he has shown in the most incontrovertible mannar, that the principles on which the 'New American Practical Navigator' is founded, are universally false, and gross impositions on the public."

It may be remarked here that these Editors, being of Hibernian descent, were prejudiced against American authors. The two authors last mentioned, the one by his "Practical Navigator," or "Epitome," and the other by his "American Coast Pilot," have done more for the interests of sea-faring men than all other authors combined. The former in particular, Nathaniel Bowditch LL. D., for years before his death, stood at the head of American Mathematicians.

Mr. A. Rabbit, on p. 142, says: "The contents of p. 203 of the 'Columbian Accountant,' sung to the tune of 'The *pump* of Lake Champlain' will

completely exhibit the stupidity of Shepherd's rule mentioned in the quest." And in a foot-note he says; "The theological properties of this wonderful pump have, I understand, been sufficiently investigated by the members of a certain church in New York, and I promise to unfold the mathematical principles of the same in some future number."

It is difficult at this late day to see the point of all this, but it probably is a covert sneer at Mr. Shepherd and his friends.

The editor, on p. 154, says; "A. Rabbit will not in any future number, be permitted to propose questions concerning the blunders of stupid Shepherds; we had rather soar aloft with the eagle, than waddle in mud with a goose."

On p. 174, it is stated by the sub-editors, that "the *health* of Mr. Baron, our principal editor, was, last summer, entirely destroyed by three of the understrappers of the *Health Committee*. Deprived of his assistance, we earnestly solicit our contributors to endeavor to render their solutions as perfect as possible." The cause assigned for the retirement of Mr. Baron from his position is as enigmatical as the riddle of the Sphynx, or a Delphic Oracle.

At this time many of the subscribers neglected to pay, and the editors threatened to publish their names, but the threat was not carried into execution, and the paper soon died out. The principal contributors were, besides Mr. Baron, Robert Adrain, Rev. T. P. Irving, Wm. Lenhart, John Smithis, Thomas Maughan, John D. Craig, John Capp, Diarius Yankee, N. Young, Walter Folger Jr., James Temple, Ebenezer R. White, Richard Tagart, &c.

The next periodical was "The Analyst, or Mathematical Museum." It was edited by Robert Adrain A. M., Professor of Mathematics and Natural Philosophy in Columbia College, New York. It appears to have been published both at Philadelphia and New York. The first number was issued in 1808. Five numbers were published; how often does not appear. The writer has never seen this periodical and the above are all the particulars he has been able to collect. It appears however that, besides the Editor, Nathaniel Bowditch, Alexander M. Fisher, Melatiah Nash, William Brown, (and probably others who had been contributors to the "Math'l Correspondent") were among the contributors to "The Analyst."

In 1818, Mr. William Marrat became editor of "The Scientific Journal," which was published at Perth Amboy, N. J., in monthly numbers; nine numbers are all that are known to have been issued, seven continuously, in 1818, from February to August inclusive, and two in 1819, viz., in July and October. The above are all the particulars that the writer has been able to collect in regard to this periodical.

The next year, 1820, Melatiah Nash became editor of "The Ladies and Gentlemen's Diary, or United States Almanac, and Repository of Science and Amusement." It was published in New York, annually of course. Besides the Almanac it contained an Ephemeris of the Sun, Moon and Planets, much valuable information in Astronomy and Philosophy, Enigmas, Charades, Rebuses, Queries and Mathematical Problems to be answered in the succeeding numbers. From a statement on pp. 62, No. I. and 77, No. III. it appears that the reason Mr. Marrat discontinued "The Scientific Journal" was, his leaving the United States, for Liverpool, England. In the same number he unites with Dr. Adrain, and others, in commending the enterprise of Mr. Nash to the patronage of the public. Only three nos. were issued. Severe and long continued sickness in his family compelled Mr. Nash to abandon the enterprise. This result was much regretted by the Scholars of that day; for he was an able man, in fact, the right man in the right place. The chief contributors to the mathematical department were Wm. Marrat of Liverpool, Eng., Dr. Bowditch, Walter Folger, John Macauley of Liverpool, J. H. Swale of do. John Capp, John D. Craig, Mr. Nash, the Editor, John Gough and Thomas Gaskin of Kendal, Eng. The latter was a youth of eleven years and a prodigy in mathematical science, considering his age.

In 1825, three years after the discontinuance of Nash's Diary, Robert Adrain LL. D., again appears as an Editor of a Mathematical periodical. He was then Professor of Mathematics and Natural Philosophy in Columbia College, N. Y., as he had been for many years. He was Editor of "The Mathematical Diary," which was, for the first two years, pnblished quarterly, and for the five succeeding years, annually. Thirteen numbers were issued; the last being a double number, so that in fact there were fourteen numbers. The first number of "The Diary" was issued Jan., 1825, and the last, March, 1832. This number contains an excellent likeness of Joseph Louis LaGrange, and an interesting sketch of his life.

Dr. Adrain continued in the editorship but one year. He had accepted the position of Professor in Rutger's College, New Brunswick, N. J., and was therefore obliged to retire from the Diary. His successor was James Ryan A. M., the author of several valuable mathematical works. For six years he ably conducted The Diary, and he would have continued to do so for many years longer, but for an unfortunate quarrel among the mathematicians. Mr. Samuel Ward 3rd, a recent graduate of Columbia College, had in part the management of the last number. In it he caused to be inserted a Dialogue, written by himself, wherein he exhibits in a ridiculous light, Dr. Henry J. Anderson, then Mathematical Professor in Columbia College.

It contains also short notices of recent mathematical works. Both the Editor and Dr. Anderson were highly indignant at this performance. The parties met at the mathematical book-store of James Ryan, and high words passed between the parties and their friends. The result was the complete breaking up of "The Diary," which was probably not intended nor anticipated by Mr. Ward. This gentleman was afterward the editor of "Young's Algebra," to which he made many important additions; and he also corrected several errors which had crept into the London edition. He was a good mathematician. He now resides in the City of Washington, and follows a line of business which is wholly different from his former pursuits. He is known there as the "King of the Lobby;" and as plain Samuel Ward, or Uncle Sam (as he is usually called). He gives the best dinners of any man in America. Not long since, on being subpoenaed to appear before the Pacific Mail Investigation Committee, he made an elaborate defense of the "Lobby;" and his arguments were most of them good and well sustained, but he took care to evade a discussion of the *shady* side of the subject. On being questioned about his splendid dinners, and their bad effects, he declared that, without a good dinner, a man could not say his prayers before retiring to rest; but, after partaking of such a dinner, he could go to bed and sleep like an angel.

Mr. Ward's father and grandfather, bearing the same name, were members of the once famous firms of "Prime and Ward," and "Prime Ward and King," the Wall street brokers.

All the most eminent mathematicians of America were contributors to "The Diary." Among them were Dr. Adrain the Editor, Dr. Bowditch, Prof. Theodore Strong, of Hamilton College, N. Y., Eugene Nulty, Benjamin Peirce, Benjamin Hallowell,(a Quaker, and a man as genial and full of humor as Samuel Ward), Marcus Catlin, J. H. Swale of Liverpool, Mathew Collins of Limerick, William Lenhart, Mary Bond of Fredricktown, Md., the only female contributor, O. Root, M. O. Shannessy A. M., John Capp, Dr. Henry J. Anderson, Charles Avery, and others.

"The Mathematical Diary" contained, besides solutions of problems, many important and valuable essays on the various branches of abstract science, and was the best mathematical serial that had as yet appeared.

The next periodical was "The Mathematical Miscellany" Charles Gill, Professor of Mathematics in St. Paul's Collegiate Institute, Flushing, Long Island, was the Editor, and it was published semi-annually at the Institute. Eight numbers were published, the first in February, 1836, and the last on Nov. 1st., 1839. It had a Junior and a Senior department. The former was for young students, and the latter was for proficients in Mathematics.

This periodical was ably conducted. The editor himself was one of the best Diophantists in America. His speculations on problems relating to Polygonal numbers were profound and interesting. Wm. Lenhart, who was also a Contributor to "The Mathematical Correspondent" furnished many original articles on the Diophantine Analysis. Professors Peirce and Strong gave integral answers to problems of the form $ax^2 + bx + c = \square$. They make use of the principle of Congruous Numbers, discovered by M. Gauss. Their solutions are the only examples of his method to be found in any American periodical. There are also a great many solutions of problems in all branches of mathematics, by the most distinguished scholars in America, among whom may be mentioned Profs. Theodore Strong LL. D. and Benj. Peirce, Professors Charles Avery and Marcus Catlin of Hamilton College, Clinton, N. Y., Messrs. George R. Perkins, O. Root, Wm. Lenhart, Lyman Abbot jr., Farrand N. Benedict, Gerardus B. Docharty, and others.

The next mathematical periodical was "The Cambridge Miscellany of Mathematics, Physics, and Astronomy," Edited by Prof. Benjamin Peirce and Joseph Lovering, Harvard University, and published quarterly. Four numbers were issued, the first in July, 1842. The writer has not seen this periodical, but it was undoubtedly conducted with the ability to be expected from the reputation of the Editors.

After an interval of fifteen years, J. D. Runkle A. M. of the Nautical Almanac office, Boston, proposed to establish a mathematical periodical on a new plan, which was approved by all the principal Educators and Mathematicians in the United States. This periodical was named "The Mathematical Monthly." The first number was issued Oct., 1858, and it was continued for three years. It was discontinued after Sept., 1861, by reason of the Civil war, which diverted the minds of men, in a great measure, from abstract subjects to the more pressing matters relating to the war.

The chief points of the plan upon which this periodical was conducted were: The publication of five problems in each number, adapted to the capacities of the younger students, to be answered in the third succeeding no. The insertion of Notes and Queries, short discussions and artcles of a fragmentary character, too valuable to be lost; and lastly, Essays not exceeding 8 pages, on various subjects, in all departments of mathematics. Besides, there were Notices and Reviews of the mathematical works issued, both old and new. Among the most interesting articles are the Account of the Comet of Donati, with elegant descriptive plates, written by the Astronomical Profesor of Harvard University, Vol. I. Nos. 2 and 3. A complete catalogue of the writings of Sir John Herschel, Vol. III. No. 7. Articles on the Indeterminate Analysis, by Rev. A. D. Wheeler of Brunswick, Maine, Vol. II. Nos. 1, 6, 12, and on the Diophantine Analysis, Vol. III. No. 11:

Other articles on the Diophantine Analysis, by Mr. Wheeler, would have been inserted if the "Mathematical Monthly" had been continued: "The Economy and Symmetry of the Honey-bees' Cells," by Chauncey Wright, Vol. II. No. 9. Simon Newcomb gives several interesting "Notes on Probabilities. In Vol. II. No. 2, there is an article containing a complete list of the writings of Nathaniel Bowditch LL. D. accompanied with short sketches of the same, which is extremely interesting. Several articles and Reviews on the Hindoo Algebra are worthy of notice. There is also an exceedingly interesting and valuable article on the "Theorem of Pappus," which is proved to be incorrectly called the Theorem of Guldinus, by J. B. Henck. There are also many other valuable articles. This periodical is embellished by portraits of Nathaniel Bowditch LL. D., Prof. Benjamin Peirce and Sir John Herschel, which are finely executed.

Besides the periodicals above mentioned, there have been several minor works published; one of these, "The Schoolmaster," by Rev. Timothy Clowes D. D., and issued at Hempsted, L. I., in 1832. It continued only a year, and was noted chiefly by a controversy between two belligerent mathematicians, both of them now living, one in New York, and the other in Taunton, Mass. It was "nip and tuck" for a long time; but at length the "Taunton man" proposed a problem which the other could not solve, and thus he bore away the palm. They are now old men, and if this notice shouid fall under their observation, the contest of 43 years ago will readily be called to mind.

There have also been Almanacs issued in past times as well as at present, containing problems and solutions. The oldest of these is "Thomas' Almanac," published at Worcester, Mass., which has existed for more than 100 years. For many years it has contained problems to be solved in the year succeeding. "The Maine Farmer's Almanac," published at Hallowell Me., is of the same character. It was edited at first and for many years by Daniel Robinson. It was founded in 1818. In the same year, "Hutchins' Improved Family Almanac" was founded; David Young being Editor. After his death, Dr. Samuel H. Wright assumed the duties of Editor, and still remains such. He also edited "The Farmers' Almanac," and "The Knickerbocker Almanac." The Lodi Manufacturing Co. also pnblished a "Farmers' Almanac." "The Antimasonic Almanac" was commenced by Edward Giddins, in 1828, at Rochester, N. Y. How long it continued the writer does not know. There are probably many other Almanacs unknown to the writer. All of these serials had more or less problems and solutions some of them problems only. It ought to have been noticed in its proper place, that John D. Williams became editor af the "Math'l Companion" in 1828, and continued it 4 years. It consisted wholely of prob's & solutions.

This periodical was evidently got up as a rival of The Mathematical Diary. His opponents were numerous, and the contest was carried on with some bitterness, till finally Mr. Williams issued his 14 famous "Challenge Problems," directed against all the mathematicians in America, excepting only Dr. Bowditch, Prof. Strong and Eugene Nulty. Six of these problems are impossible. Some of the others are somewhat difficult, but have all been solved by several persons.

Besides the ANALYST, for which this paper is a contribution, and which is devoted exclusively to mathematics, there are many periodicals at present in America which appropriate a portion of their space to mathematics.

"The Yates Co. Chronicle," a newspaper published weekly at Penn Yan, Yates Co. N. Y., by S. C. and E. Cleveland, has a mathematical department edited by Dr. Samuel H. Wright, containing problems and solutions in nearly every branch of mathematical science and is undoubtedly the best of its kind in the country. "The Railroad Gazette," published weekly at N. Y. and Chicago, has mathematical problems relating to the construction of Railroads, Engines, Cars, &c. There are also several monthly Educational periodicals which have a department devoted to mathematics. "The Schoolday Magazine," published at Philadelphia has a mathematical department which is ably edited by Artemas Martin Esq., of Erie, Pa. It has done much good to the class of advanced students for whom it was intended. "The Normal Monthly," edited by Prof. Edward Broaks at Millersville, Lancaster Co. Pa. has a mathematial department in which Artemas Martin has a series of articles on the Diophantine Analysis, which he well illustrates and in a style adapted to general comprehension. "Educational Notes and Queries," edited by Hon. W. D. Henkle at Salem, Ohio, has also a department for mathematical Notes and Queries which bids fair to be interesting and useful. There are also the "Illinois Schoolmaster," and Morton's Monthly, published at Chicago and Louisville respectively, which contain many excellent solutions of mathematical problems.

The above are all the serials, having a mathematical department which have come under the notice of the writer.

NOTE ON THE REACTIONS OF CONTINUOUS BEAMS.

BY MANSFIELD MERRIMAN, C. E., NEW HAVEN, CONN.

As a matter of purely mathematical interest I wish to give here, without demonstration, the relations between the reactions of continuous girders of equal spans resting on level supports.

Let a girder of s equal spans be subjected to any number of concentrated loads, and the distance of any load P from the nearest left hand support be denoted by kl, l being the length of a span; let P_1, P_2, P_r etc., denote concentrated loads on the 1st, 2nd, rth, etc., spans, and R_1, R_2, R_r, etc., the reactions at the points of support. Then the relation between the reactions is given by the equations,

$$6R_1 + R_2 = \Sigma P_1(6 - 6k + k^3) + \Sigma P_2(1 - k)^3$$
$$4R_2 + R_3 = \Sigma P_1(6k - 2k^3) + \Sigma P_2(4 - 6k^2 + 3k^3) + \Sigma P_3(1 - k)^3$$
$$R_2 + 4R_3 + R_4 = \Sigma P_1 k^3 + \Sigma P_2(1 + 3k + 3k^2 - k^3) + \Sigma P_3(4 - 6k^2 + 3k^3) + \Sigma P_4(1 - k)^3$$

* * * * * * * * * * * *

$$R_{r-1} + 4R_r + R_{r+1} = \Sigma P_{r-2} k + \Sigma P_{r-1}(1 + 3k + 3k^2 - k^3) + \Sigma P_r(4 - 6k^2 + 3k^3) + \Sigma P_{r+1}(1 - k)^3$$

* * * * * * * * * * * *

$$R_{s-1} + 4R_s = \Sigma P_{s-2} k^3 + \Sigma P_{s-1}(1 + 3k + 3k^2 - 3k^3) + \Sigma P_s(4 - 6k^2 + 2k^3)$$
$$R_s + 6R_{s+1} = \Sigma P_{s-1} k^3 + \Sigma P_s \ (1 + 3k + 3k^2 - \ k^3)$$

If the spans be uniformly loaded throughout with loads w_1, w_2, w_r etc., per unit of length, these equations become,

$$6R_1 + R_2 = \tfrac{13}{4} w_1 l + \tfrac{1}{4} w_2 l$$
$$4R_2 + R_3 = \tfrac{10}{4} w_1 l + \tfrac{11}{4} w_2 l + \tfrac{1}{4} w_3 l$$
$$R_2 + 4R_3 + R_4 = \tfrac{1}{4} w_1 l + \tfrac{11}{4} w_2 l + \tfrac{11}{4} w_3 l + \tfrac{1}{4} w_4 l$$
$$\text{etc.} \qquad \text{etc.} \qquad \text{etc.}$$
$$R_s + 6R_{s+1} = \tfrac{1}{4} w_{s-1} l + \tfrac{13}{4} w_s l,$$

and for a uniform load of w per linear unit over the whole beam, they are

$$6R_1 + R_2 = \tfrac{7}{2} wl$$
$$4R_2 + R_3 = \tfrac{11}{2} wl$$
$$R_2 + 4R_3 + R_4 = 6wl$$
$$\text{etc.} \qquad \text{etc.}$$

From the time of Navier to the discovery by Clapeyron of the Theorem of Three Moments, the method of investigation of continuous girders consisted in first determining, by long and tedious equations, the reactions at the supports, and then from there deducing the shears and moments for any required section. This was undoubtedly the most logical method—to find all the exterior forces under which the beam was held in equilibrium and then pass to the internal strains. But so long and difficult was the labor of finding the reactions that the theory was but slightly advanced until the happy discovery of Clapeyron of the relation existing between the moments at the points of support. This Theorem with its later extension to concentrated loads and variable moments of inertia serves as a starting point from

which the whole theory is easily deduced and put in to shape for practical use.*

Those familiar with the history of this subject before the time of Clapeyron's discovery will at once recognize the simplicity of the above equations compared with the complicated method of Navier.

NOTE ON THE DIVISION OF SPACE.

BY PROF. HALL.

This question occurred to me several years ago in reading an account of Bertrand's method of treating the doctrine of parallel lines. I have not seen Prof. Cayley's solution, but of course so obvious a question could not be new. Lately I have found that this question and several kindred ones are very completely discussed by Steiner in the first Vol. of Crelle's Journal, published in 1826. For the number of parts into which n planes can divide space Steiner finds an expression which is eqnivalent to $\frac{1}{6}(n^3+5n+6)$; and he shows also that of these parts

$$\frac{(n-1)(n-2)(n-3)}{1\,.\,2\,.\,3} \text{ are limited,}$$

and $\quad 2+n(n-1)$ are unlimited.

CORRECTION OF AN ERROR IN BARLOW'S THEORY OF NUMBERS.

BY ARTEMAS MARTIN, ERIE, PA.

In *Barlow's Theory of Numbers*, page 299, it is stated that "the equation

$$x^2-5658y^2=1$$

has its least values as follows; viz.

$$x=166100725257977318398207998462201324702014613503,$$
$$y=\quad 698253616416770487157775940222021002391003072."$$

I will show that these values are not correct, and then compute the true numbers.

The units figure of the square of Barlow's value of x is 9; the units figure of the square of his value of y is 4; the units figure of $5658y^2$ is 2, and 9 — 2=7. If his values were correct the units figure of x^2 would be 1 greater

*For a full presentation of ready methods for finding moments and reactions the reader may consult the *Journal of the Franklin Institute*, March and April, 1875.

than the units figure of $5658y^2$ unless the units figure of x^2 was 0, in which case the units figure of $5658y^2$ would be 9.

Let $A = 5658$, then $\sqrt{(5658)} = \sqrt{A} = r + \cfrac{1}{u_1 + \cfrac{1}{u_2 + \cfrac{1}{u_3 + \&c.,}}}$

where r is the greatest integer contained in $\sqrt{A}$.

The last quotient of every complete period is $2r$. Let m be the number of quotients in a complete period, and $p_m \div q_m$ the last convergent in the first period; then, when m is even, $x = p_m$, $y = q_m$, and when m is odd $x = p_{2m}$, $y = q_{2m}$.

Let $\dfrac{\sqrt{(A)} + a_n}{b_n} = u_n + \&c.$ and $\dfrac{\sqrt{(A)} + a_{n+1}}{b_{n+1}} = u_{n+1} + \&c.$ be any two consecutive complete quotients, then

$$a_{n+1} = u_n b_n - a_n, \quad b_{n+1} = \frac{A - a_{n+1}^2}{b_n}.$$

If $p_n \div q_n$, $p_{n+1} \div q_{n+1}$ be any two consecutive convergents and u_{n+1} the quotient corresponding to $p_{n+1} \div q_{n+1}$, then

$$\frac{p_{n+2}}{q_{n+2}} = \frac{u_{n+1} p_{n+1} + p_n}{u_{n+1} q_{n+1} + q_n}.$$

$$\frac{\sqrt{(5658)} + 0}{1} = 75 + = r, \qquad \frac{r + 75}{33} = 4 + = u_1,$$

$$\frac{r + 57}{73} = 1 + = u_2, \qquad \frac{r + 16}{74} = 1 + = u_3,$$

$$\frac{r + 58}{31} = 4 + = u_4, \qquad \frac{r + 66}{42} = 3 + = u_5,$$

$$\frac{r + 60}{49} = 2 + = u_6, \qquad \frac{r + 38}{86} = 1 + = u_7,$$

$$\frac{r + 48}{39} = 3 + = u_8, \qquad \frac{r + 69}{23} = 6 + = u_9,$$

$$\frac{r + 69}{39} = 3 + = u_{10}, \qquad \frac{r + 48}{86} = 1 + = u_{11},$$

$$\frac{r + 38}{49} = 2 + = u_{12}, \qquad \frac{r + 60}{42} = 3 + = u_{13},$$

$$\frac{r + 66}{31} = 4 + = u_{14}, \qquad \frac{r + 58}{74} = 1 + = u_{15},$$

$$\frac{r + 16}{73} = 1 + = u_{16}, \qquad \frac{r + 57}{33} = 4 + = u_{17},$$

$$\frac{r + 75}{1} = 150 + = u_{18} = 2r.$$

As 18, the number of quotients in a complete period, is even, therefore $x = p_{18}$ and $y = q_{18}$.

$\frac{p_1}{q_1} = \frac{75}{1}$, $\frac{p_2}{q_2} = \frac{301}{4}$, $\frac{p_3}{q_3} = \frac{376}{5}$, $\frac{p_4}{q_4} = \frac{677}{9}$, $\frac{p_5}{q_5} = \frac{3084}{41}$, $\frac{p_6}{q_6} = \frac{9929}{132}$, $\frac{p_7}{q_7} = \frac{22942}{305}$, $\frac{p_8}{q_8} = \frac{32871}{437}$, $\frac{p_9}{q_9} = \frac{121555}{1616}$, $\frac{p_{10}}{q_{10}} = \frac{762201}{10133}$, $\frac{p_{11}}{q_{11}} = \frac{2408158}{32015}$, $\frac{p_{12}}{q_{12}} = \frac{3170359}{42148}$, $\frac{p_{13}}{q_{13}} = \frac{8748876}{116311}$, $\frac{p_{14}}{q_{14}} = \frac{29416987}{391081}$, $\frac{p_{15}}{q_{15}} = \frac{126416824}{1680635}$, $\frac{p_{16}}{q_{16}} = \frac{155833811}{2071716}$, $\frac{p_{17}}{q_{17}} = \frac{282250635}{3752351}$, $\frac{p_{18}}{q_{18}} = \frac{1284836351}{17081120}$.

$\therefore \quad x = 1284836351, \quad y = 17081120.$

I have verified these numbers.

NOTE ON THE SOLUTION OF MR. HOLBROOK'S QUESTION.

BY THE EDITOR.

Prof. Eddy (see p. 126) has given the equation of a solid whose surface is generated as described in the question proposed by Mr. Holbrook on page 72; but it appears from a subsequent clause that Mr. Holbrook had a different question in view, viz.; he asks for a "demonstration that no two ellipses can be parallel."

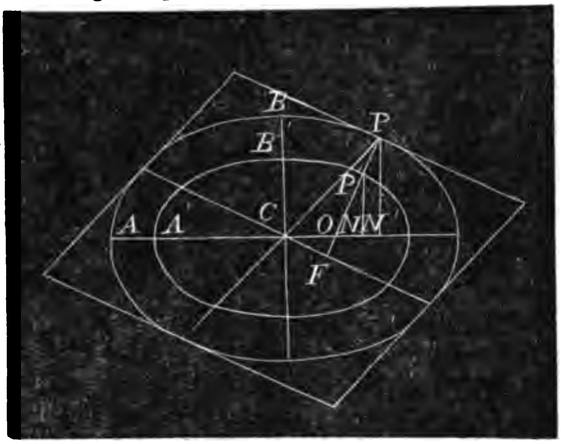
The surface of a solid with an elliptical base, horizontal sections of which shall be bounded by curves parallel to the periphery of the base, may be generated by a straight line which makes a constant angle with the *normal* to the ellipse while the extremity of the line describes the periphery of the ellipse. That no horizontal section, above the base of such a solid, can be an ellipse, is what we understand Mr. Holbrook to assert and desire to have demonstrated.

Horizontal sections of the solid of which Prof. Eddy has given the equation, are obviously ellipses; but that no section, above the base, of the solid whose surface is generated as above described, can be an ellipse, may be demonstrated as follows:

Let ABP (see diagram on next page) represent an ellipse whose semi-axes are $AC = a$ and $BC = b$, and the normal of which, at any point P, is $PO = N$. Let $A'B'P'$ be a parallel curve within the ellipse; then will the portion PP' of the normal, be of the same length for all points of the ellipse. We may therefore put $PP' = c$, a constant.

By property 12 of the ellipse, (see Bridge's Conic Sections), we have $PO \times PF = BC^2$. And, assuming the parallel curve $A'B'P'$ to be an ellipse, we have, by the same property, $P'O \times P'F = B'C^2$. By substituting for $P'O$, $P'F$ and $B'C$ their equivalents $PO - c$, $PF - c$, and $BC - c$, we get $PO + PF = 2BC$, from which, by squaring both sides we get $PO^2 + 2PO \times PF + PF^2 = 4BC^2$; but $2PO \times PF = 2BC^2$; $\therefore$ by subtraction we get

$$PO^2 + PF^2 = 2BC^2.$$



Because $2BC^2 = 2PO \times PF$, therefore $PO^2 + PF^2 = 2PO \times PF$. But this equation can only be true when $PO = PF$, and in that case the curve ABP is a circle. Hence our assumption that the parallel curve is an ellipse, is not true; and, therefore, no curve that is parallel to an ellipse can be an ellipse.

The equation of the parallel curve may be found as follows: Let x, y be rectangular coordinates to any point P of the ellipse, the origin being at the center, and N the normal at the point P. Also, let x', y' be rectangular coordinates and N' the normal to the corresponding point P' of the parallel curve $A'B'P'$; and let a, b be the semi-axes of the ellipse and a', b' the semi-axes of the parallel curve, then is

$$a = a' + c,\ b = b' + c, \text{ and } N = N' + c, \ldots\ldots \quad (1)$$

and from similar triangles we find $y = \dfrac{N' + c}{N'} y'$. . . . . (2)

The equation for the length of the normal is readily found to be

$$N = y\sqrt{\left(1 + \frac{dy^2}{dx^2}\right)}. \ldots\ldots\ldots \quad (3)$$

Differentiating the equation to the ellipse, we find

$$\frac{dy^2}{dx^2} = \frac{b^4 - b^2y^2}{a^2y^2}.$$

Therefore, by substitution, $N = \dfrac{1}{a}\sqrt{\left[(a^2 - b^2)y^2 + b^4\right]}$. . . . . (4)

Substituting in (4) from (1) and (2) we get, for the required equation,

$$y' = \frac{N'}{N' + c}\sqrt{\left[\frac{[(N' + c)(a' + c)]^2 - (b' + c)^4}{(a' + c)^2 - (b' + c)^2}\right]}.$$

PROOF THAT NO TWO DIFFERENT ELLIPSES CAN BE PARALLEL.

PY PROF. E. W. HYDE, CINCINNATI, OHIO.

The equation of an ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

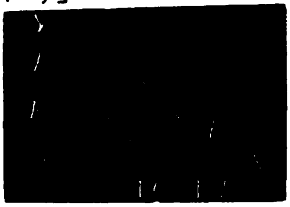
Therefore, if there can be a parallel ellipse its equation must be of the form

$$\frac{x^2}{(a+c)^2}+\frac{y^2}{(b+c)^2}=1;$$

i. e. the same quantity c is added to each semi-axis. Now if these curves are parallel the distance between a tangent to one and a parallel tangent to the other on the same side of the origin must be equal to c. The equations of a pair of such parallel tangents are

(1) $$y=mx+\sqrt{(a^2m^2+b^2)} \quad \text{and}$$

(2) $$y=mx+\sqrt{[(a+c)^2m^2+(b+c)^2]}.$$



The distance between the two tangents in the figure is

$$CE=CD\sin(\tan^{-1}m)=(OD-OC)\frac{m}{\sqrt{(1+m^2)}}$$

For convenience let the tangents be inclined at an angle of 135°, i. e. let $m=-1$; then by making $y=0$ in (1) and (2) $\quad OC=\sqrt{(a^2+b^2)}$

and $\quad OD=\sqrt{[(a+c)^2+(b+c)^2]}.$

$$\therefore \quad CE=\sqrt{\tfrac{1}{2}}\times\{\sqrt{[(a+c)^2+(b+c)^2]}-\sqrt{(a^2+b^2)}\}.$$

When $b=a$, $CE=c$ as it should; but if b is different in value from a, then CE will be different in value from c, and the two curves cannot be parallel. To prove this, place

$$\sqrt{\tfrac{1}{2}}\times\{\sqrt{[(a+c)^2+(b+c)^2]}-\sqrt{(a^2+b^2)}\}=c.$$

Transposing, squaring and cancelling we obtain finally $a=b$, showing that this condition is *necessary* in order that the equation may be true.

[We have also received a paper from Prof. Hyde supplimentary to his solution of Mr. Church's problem, (p. 76), in which he effects a solution of the last part of that problem, or rather of the "secondary" problem to which he has reduced that problem by projection, in a more simple and elegant manner. We have also received solutions of the same problem from Prof. Johnson and Dr. Eggers which we hope, at some future time, to present to our readers.—Ed.]

THE POTENTIAL FUNCTION.

BY PROF. DAVID TROWBRIDGE, WATERBURGH, N. Y.

1. On p. 100, Vol. I. of THE ANALYST, Eq. (11), I have given the following value of the Potential Function:

$$V=\iiint\frac{\rho r'^2\sin\theta' dr' d\theta' d\omega'}{\{r^2+r'^2-2rr'[\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\omega-\omega')]\}^{\frac{1}{2}}};\quad(1)$$

the integrations extending over the entire attracting mass. Now let

$$r'=cr,\ \text{and}\ p=\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\omega-\omega'),\ \ldots\ldots\ (2)$$

and

$$R=(1+c^2-2cp)^{-\frac{1}{2}},\ \ldots\ldots\ldots\ldots\ (3)$$

then

$$V=\iiint\frac{1}{r}\rho R r'^2\sin\theta' dr' d\theta' d\omega'.\ \ldots\ldots\ldots\ (4)$$

Let R be developed in to the following series:

$$R=1+cP_1+c^2P_2+c^3P_3+\ldots+c^iP_i+\ldots\qquad(5)$$

P_i is a function of p independent of c. If we take the partial differential coefficients of (3) with respect to p we shall have

$$D_pR=c(1+c^2-2cp)^{-\frac{3}{2}}=cR^3,\ D_p^2R=3cR^2D_pR=3c^2R^5,\ D_p^3R$$
$$=3\,.\,5c^3R^7,\ D_p^4R=3\,.\,5\,.\,7c^4R^9,\ldots\ D_p^nR=3\,.\,5\,.\,7\ldots(2n-1)c^nR^{2n+1}.\ (6)$$

Now let $R^{2n+1}=1+cP_1^{(n)}+c^2P_2^{(n)}+\ldots+c^iP_i^{(n)}+\ldots\qquad(7)$

If we take the partial differential coefficients of (5) with respect to p, multiply (7) by $3\,.\,5\,.\,7\ldots(2n-1)c^n$, equate the coefficients of like powers of c, we shall find

$$D_p^nP_i=1\,.\,3\,.\,5\,.\,7\ldots(2n-1)P_{i-n}^{(n)}.\ \ldots\ldots\ldots\ (8)$$

From this equation we can find $P_1^{(n)}$, $P_2^{(n)}$, &c. when P_1, P_2, &c. are known in terms of p.

2. From (3) we find, by taking partial differentials,

$$D_cR=-(c-p)R^3,\ D_c(c^2D_cR)=-D_c(c^3-c^2p)R^3=-(3c^2-2cp)R^3$$
$$-3(c^3-c^2p)R^2D_cR=-(3c^2-2cp)R^3+3c^2(c-p)^2R^5$$
$$D_pR=cR^3,\ D_p(p^2D_pR)=cD_p(p^2R^3)=2cpR^3+3c^2p^2R^5,\ D_p^2R=3c^2R^5.$$

From these equations we find, since $R^2(1+c^2-2cp)=1$,

$$D_c(c^2D_cR)+D_p^2R-D_p(p^2D_pR)=-c(3c-2p)(1+c^2-2cp)R^5$$
$$+3c^2(c-p)^2R^5+3c^2R^5-2cp(1+c^2-2cp)R^5-3c^2p^2R^5=0.$$

$$\therefore\quad D_c(c^2D_cR)+D_p[(1-p^2)D_pR]=0.\ \ldots\ldots\ldots\ (9)$$

If we substitute the value of R given by (5) in this equation, and equate the coefficients of like powers of c, we shall find for the general value

$$i(i+1)P_i + D_p[(1-p^2)D_pP_i] = 0. \quad \ldots\ldots (10)$$

By means of this equation we are to calculate the value of P_i. From (3) and (5) we have

$$D_cR = -\frac{c-p}{(1+c^2-2cp)^{\frac{3}{2}}} = P_1 + 2P_2c + \ldots + iP_ic^{i-1} + \ldots$$

$$(p-c)(1 + P_1c + P_2c^2 + \ldots + P_{i-1}c^{i-1} + \ldots) = (1+c^2-2cp) \times (P_1 + 2P_2c + \ldots iP_ic^{i-1} + \ldots).$$

From this equation we have

$$iP_i = (2i-1)pP_{i-1} - (i-1)P_{i-2}. \quad \ldots\ldots (11)$$

If $i=2$, since $P_0 = 1$, and $P_1 = p$, $2P_2 = 3p^2 - 1$; and for $i = 3$, $3P_3 = 5pP_2 - 2P_1 = \frac{1}{2}(5p^3 - 3p)$.
From these equations we readily see that P_i will have this form

$$P_i = B_0p^i + B_1p^{i-2} + B_2p^{i-4} + \ldots + B_sp^{i-2s} + \ldots \quad (12)$$

If we substitute the value of P_i given by (12) in (10) and equate the coefficients of like powers of p, we shall have

$$i(i+1)B_s + (i-2s+2)(i-2s+1)B_{s-1} - (i-2s)(i-2s+1)B_s = 0.$$

$$\therefore \quad B_s = -\frac{(i-2s+2)(i-2s+1)}{2s(2i-2s+1)}B_{s-1}. \quad \ldots\ldots (13)$$

Now make $s = 1, 2, 3,$ &c. in succession, and

$$B_1 = -\frac{i(i-1)}{2(2i-1)}B_0,\ B_2 = -\frac{(i-2)(i-3)}{4(2i-3)}B_1 = \frac{i(i-1)(i-2)(i-3)}{2.4(2i-1)(2i-3)}B_0$$

$$\text{\&c.} \quad \ldots\ldots (14)$$

We can in this way find all the coefficients in terms of B_0. We see that B_0 is the coefficient of c^ip^i in the development of R. We have

$$R = (1+c^2)^{-\frac{1}{2}}\left(1 - \frac{2cp}{1+c^2}\right)^{-\frac{1}{2}} = (1+c^2)^{-\frac{1}{2}}\left[1 + \frac{2\lambda p}{2} + \frac{3.2^2\lambda^2p^2}{2.4} \ldots + \frac{3.5\ldots(2i-1)2^i\lambda^ip^i}{2.4.6\ldots 2i} + \ldots\right].\quad \lambda = \frac{c}{1+c^2}$$

From this we see that the coefficient of c^ip^i is $\dfrac{1.3.5\ldots 2i-1}{1.2.3\ldots i} = B_0 . . (15)$

$$\text{Hence } P_i = \frac{1.3.5\ldots 2i-1}{1.2.3\ldots i}\left[p^i - \frac{i(i-1)}{2(2i-1)}p^{i-2} + \frac{i(i-1)(i-2)(i-3)}{2.4(2i-1)(2i-3)}p^{i-4} - \ldots\right] \quad \ldots\ldots (16)$$

If we put aP_i for P_i in (10) the equation will still be satisfied; so that so long as a function of p differs from P only by having a constant multiplier, greater or less than unity, it will satisfy (10). The quantity P_i I shall call the p—coefficient of the ith order; and any other quantity as F_i, that will satisfy (10), I shall call the p—function of the ith order, i being any integer which denotes the highest power of p that enters in to the *coefficient* or the *function.*

If we should substitute for p its value given by (2), equation (10) would then be known as Laplace's Equation; and P_i would be called Laplace's Coefficient of the ith order; and F_i would be called Laplace's Function.

3. Let P_i and Q_i be any two p—functions. Equation (10) gives

$$i(i+1)\int_{-1}^{+1} P_i Q_n dp = -\int_{-1}^{+1} Q_n D_p[(1-p^2)D_p P_i]dp \quad \ldots . (17)$$

$$n(n+1)\int_{-1}^{+1} P_i Q_n dp = -\int_{-1}^{+1} P_i D_p[(1-p^2)D_p Q_n]dp \quad \ldots . (18)$$

$$i(i+1)\int_{-1}^{+1} P_i Q_n dp = -\Big[_{-1}^{+1} Q_n[(1-p^2)D_p P_i]\Big] + \int_{-1}^{+1} D_p P_i[(1-p^2)D_p Q_n]dp$$

$$= \Big[_{-1}^{+1} P_i[(1-p^2)D_p Q_n]\Big] - \int_{-1}^{+1} P_i D_p[(1-p^2)D_p Q_n]dp = n(n+1)\int_{-1}^{+1} P_i Q_n dp,$$

by (18). Therefore $[i(i+1) - n(n+1)]\int_{-1}^{+1} P_i Q_n dp = 0.$

So long as i differs from n, $i(i+1) - n(n+1)$ is not zero and therefore

$$\int_{-1}^{+1} P_i Q_n dp = 0. \quad \ldots \ldots \ldots \ldots \ldots (19)$$

This is a very important result. If $i = n$ it is indeterminate. For the case where $i = n$ we shall proceed as follows:

If we make $X = 1 - p^2$, then $D_p[(1-p^2)D_p P_i] = D_p(XD_p P_i)$. If this be differentiated m times we shall have

$$D_p^m(XD_p P_i) = D_p^m X D_p P_i + mD_p^{m-1} X D_p^2 P_i + \ldots + \frac{m(m-1)}{1.2} D_p^2 X D_p^{m-1} P_i$$

$$+ mD_p X D_p^m P_i + XD_p^{m+1} P_i = -m(m-1)D_p^{m-1} P_i - 2mpD_p^m P_i$$

$$+ (1-p^2)D_p^{m+1} P_i, \text{ since } X = 1 - p^2.$$

If we apply this to (10), we shall have, since $i(i+1) - m(m-1)$ $= (i-m+1)(i+m)$, by multiplying the resulting equation by $(1-p^2)^{m-1}$

$$D_p[(1-p^2)^m D_p^m P_i] + (i-m+1)(i+m)(1-p^2)^{m-1} D_p^{m-1} P_i = 0 \ldots (20)$$

Now multiply (20) by $D_p^{m-1}P_n$, and integrate the first term by parts, and

$$\int_{-1}^{+1} D_p[(1-p^2)^m D_p^m P_i] D_p^{m-1} P_n dp = \Big[(1-p^2)^m D_p^m P_i D_p^{m-1} P_n\Big]_{-1}^{+1}$$

$$- \int_{-1}^{+1}(1-p^2)^m D_p^m P_i D_p^m P_n\, dp$$

$$= -\int_{-1}^{+1}(1-p^2)^m D_p^m P_i D_p^m P_n dp$$

$$= -(i-m+1)(i+m)\int_{-1}^{+1}(1-p^2)^{m-1} D_p^{m-1} P_i D_p^{m+1} P_n dp;$$

or $\displaystyle\int_{-1}^{+1}(1-p^2)^m D_p^m P_i D_p^m P_n dp$

$$= (i-m+1)(i+m)\int_{-1}^{+1}(1-p^2)^{m-1} D_p^{m-1} P_i D_p^{m-1} P_n dp. \,.(21)$$

If we now make in succession $m = 1, 2, 3,$ &c., we shall have

$$\int_{-1}^{+1}(1-p^2)D_pP_iD_pP_ndp = i(i+1)\int_{-1}^{+1}P_iP_ndp, \quad \int_{-1}^{+1}(1-p^2)^2D_p^2P_iD_p^2P_ndp$$

$$= (i-1)(i+2)\int_{-1}^{+1}(1-p^2)D_pP_iD_pP_ndp = i(i-1)(i+1)(i+2)\int_{-1}^{+1}P_iP_ndp;$$

and finally $\displaystyle\int_{-1}^{+1}(1-p^2)^m D_p^m P_i D_p^m P_n dp$

$$= (i-m+1)(i-m+2)\ldots i(i+1)\ldots(i+m)\int_{-1}^{+1}P_iP_ndp. \,.(22)$$

Now make $n = i$ and $m = i$, and we have

$$\int_{-1}^{+1}(1-p^2)^i D_p^i P_i D_p^i P_i dp = 1.2.3.4\ldots 2i\int_{-1}^{+1}P_i^2dp. \,\ldots\ldots\ldots (23)$$

If we differentiate the value of P_i given by (16), i times, we shall have

$$D_p^iP_i = 1.3.5\ldots 2i-1. \,\ldots\ldots\ldots\ldots (24)$$

This value in (23) gives

$$[1.3.5\ldots 2i-1]^2\int_{-1}^{+1}(1-p^2)^idp = 1.2.3.4\ldots 2i\int_{-1}^{+1}P_i^2dp,$$

or $\displaystyle\int_{-1}^{+1}P_i^2dp = \frac{1.3.5\ldots 2i-1}{2.4.6\ldots 2i}\int_{-1}^{+1}(1-p^2)^idp$

$$= \frac{1.3.5\ldots 2i-1}{2.4.6\ldots 2i}\cdot\frac{2.4.6\ldots 2i}{1.3.5\ldots 2i-1}\cdot\frac{2}{2i+1}.$$

$$\therefore \qquad \int_{-1}^{+1}P_i^2dp = \frac{2}{2i+1}. \,\ldots\ldots\ldots\ldots\ldots (25)$$

In this equation i is any integer. We can easily verify it for $i = 2$, since $P_2 = \frac{3}{2}(p^2 - \frac{1}{3})$, and also for $i = 3$, &c. The result expressed by (25) is also a very important one.

4. It is possible (granting that all differential equations of one variable are integrable) to arrange all algebraic functions of p, that do not become infinite between the limits of integration, into a series of p— functions, as will thus be seen. Let X be any algebraic function of p. In order that we may select from X what will make a p– function of the order i, say F_i, it is only necessary to find the coefficient which corresponds to what we have represented by B_0, (15), in the coefficients; for the *law* of the terms of the function is fixed, being the same as in (16). Let us call the required coefficient $A_0^{(2)}$, then it will be evident by (25) that

$$B_0 \int_{-1}^{+1} P_i F_i dp = \frac{2A_0^{(i)}}{2i+1}; \quad \ldots \ldots \ldots \ldots \ldots \quad (26)$$

or,

$$B_0 \int_{-1}^{+1} P_i X dp = \frac{2A_0^{(i)}}{2i+1}; \quad \ldots \ldots \ldots \ldots \ldots \quad (27)$$

since by (19) all the terms of X not required to form F_i, will disappear. If X be developed according to the positive powers of p, then we may make

$$X = A_0^{(i)}\left[p^i - \frac{i(i+1)}{2(2i-1)}p^{i-2} + \ldots\right] + A_0^{(i-1)}\left[p^{i-1} - \frac{(i-1)(i-2)}{2(2i-3)}p^{i-3} + \ldots\right] + \ldots$$

and compare the coefficients of like powers of p. Let
$X = p^3 + p^2 + p + 1 = A_0^{(0)} + A_0^{(1)}p + A_0^{(2)}(p^2 - \frac{1}{3}) + A_0^{(3)}(p^3 - \frac{3}{5}p)$.
Then $\quad A_0^{(3)} = 1, \quad A_0^{(2)} = 1, \quad A_0^{(1)} = \frac{8}{5}, \quad A_0^{(0)} = \frac{4}{3}$.

By (27) $\int_{-1}^{+1}(p^3 + p^2 + p + 1)dp = 2A_0^{(0)} = 2(\frac{1}{3} + 1)$, $A_0^{(0)} = \frac{4}{3}$,

$\int_{-1}^{+1}(p^4 + p^3 + p^2 + p)dp = \frac{2}{3}A_0^{(1)} = 2(\frac{1}{5} + \frac{1}{3})$, $A_0^{(1)} = \frac{8}{5}$,

$\frac{9}{4}\int_{-1}^{+1}(p^2 - \frac{1}{3})(p^3+p^2+p+1)dp = \frac{2}{5}A_0^{(2)} = \frac{9}{2}(\frac{1}{5}+\frac{1}{3} - \frac{1}{9} - \frac{1}{3})$, $A_0^{(2)} = 1$; &c.;

since $P_0 = 1$, $P_1 = p$, $P_2 = \frac{3}{2}(p^2 - \frac{1}{3})$, $P_3 = \frac{5}{2}(p^3 - \frac{3}{5}p)$, &c.

Since the quantity within the brackets in (16) must be the same for all these functions, it is evident that if X is a surd, as $\sqrt{(1 + p^2)}$, the number of functions is infinite. Now let $X = A_0 + A_1 p^2$, and we shall find two functions of the order 0 and 2, as follows;

$$A_0 + A_1 p^2 = (A_0 + \tfrac{1}{3}A_1) + A_1(p^2 - \tfrac{1}{3}). \quad \ldots \ldots \quad (28)$$

5. Let us now make an application of the principles which we have demonstrated, to find the potential of an oblate spheroid for an external point situated in the prolongation of the axis of revolution, the density being homogeneous.

From (2), (4), and (5) we have

$$V=\int_{-1}^{+1}\int_0^{r'}\int_0^{2\pi}\rho r'^2 dr'dpd\omega'\left[\frac{1}{r}+P_1\frac{r'}{r^2}+P_2\frac{r'^2}{r^3}+\ldots+P_i\frac{r'^i}{r^{i+1}}+\right]$$

by making $\sin\theta' d\theta' = -dp$ (as it evidently is), and changing the sign of V. Since ω' is independent of r' and p, and the first term evidently the mass of the spheroid divided by r (equal to $m\div r$,)

$$V=\frac{m}{r}+2\pi\rho\int_{-1}^{+1}dp\left[\tfrac{1}{4}P_1\frac{r'^4}{r^2}+\tfrac{1}{5}P_2\frac{r'^5}{r^3}+\ldots+\frac{1}{i+3}P_i\frac{r'^{i+3}}{r^{i+1}}+\ldots\right]\ldots(29)$$

Let $r'=\dfrac{a\sqrt{(1-e^2)}}{\sqrt{[1-e^2(1-p^2)]}}$, and let $\dfrac{r'^{i+3}}{a^{i+3}}$ be developed in to a series of p–functions, so that

$$r'^{i+3}=a^{i+3}[F_0+F_1+F_2+\ldots+F_i+\ldots]\ldots.(30)$$

If this be substituted in (29) we see by (19) that every term, when integrated, except the one containing P_i will disappear. If we expand the value of r'^{i+3}, retaining only e^2, we shall find

$$r'^{i+3}=a^{i+3}\left[1-\frac{i+3}{2}e^2p^2\right]=a^{i+3}\left[\left(1-\frac{i+3}{6}e^2\right)-\frac{i+3}{2}e^2\left(p^2-\tfrac{1}{3}\right)\right]$$
$$=a^{i+3}[F_0+F_2].$$

From this we see that $i=0$ and $i=2$ are the only values to be used; and since there is no P_0 in (29), we have

$$V=\frac{m}{r}-\pi\rho\frac{a^5}{r^3}\int_{-1}^{+1}e^2P_2(p^2-\tfrac{1}{3})dp=\frac{m}{r}-\pi e^2\rho\frac{a^5}{r^3}\int_{-1}^{+1}\tfrac{3}{2}(p^2-\tfrac{1}{3})^2dp.$$

$$\therefore\qquad V=\frac{m}{r}-\frac{4}{15}\pi e^2\rho\frac{a^5}{r^3}=\frac{m}{r}-\frac{ma^2e^2}{5r^3}.\ \ldots\ldots\ldots(31)$$

The preceding discussion will help the student to understand the nature and uses of Laplace's *Coefficients* and *Functions* in their more general form as given in works on the figure of the earth and elsewhere. Some mathematical expressions contain curious properties.

RECENT MATHEMATICAL PUBLICATIONS.

COMMUNICATED BY G. W. HILL.

Gauss, C. F. Werke. Band VI. Herausgegeben von der koniglichen Gesellschaft der Wissenschaften zu Gottingen. Gottingen. 1874. 4to. 664 pp. 25 M.

Reuschle, C. G. Tafeln complexer Primzahlen, welche aus Wurzeln der Einheit gebildet sind. Auf dem Grunde der Kummerschen Theorie der complexen Zahlen berechnet. Berlin. 1875. 4to. VII. 671 pp. 24M.

Konigsberger, Dr. Leo. Verlesungen uber die Theorie der elliptischen Functionen, nebst einer Einleitung in die allgemeine Functionenlehre. 2 Theilen. Leipzig. 1874. 8vo. 432, 219 pp.

INVENTION OF A NEW NUMERICAL SYSTEM.

BY FERDINAND EISSFELDT, BOSTON, MASS.

In our common decimal system, distinct characters are given to the numbers from one to ten; and it is very well known that instead of ten, any other number, for instance eight, or twelve or sixteen, or even two, may be selected for a base. Such systems have actually been calculated, but they have not come into use: the advantage not being sufficient to counterbalance the inconvenience of changing one system into another.

The subject may be treated, however, in quite a different manner yet. There is no necessity for taking any base at all, and the numbers may be made to progress in their own natural succession; or, to express it in other words, every number, even one, may be made to serve as a base for a certain time. This can be accomplished as follows.

Instead of dividing the original units by ten and ten, as in our common decimal system, we divide them in the natural succession of the numbers themselves, first one, then two, then three and so forth, and for each division we make a new mark in the second column. If in the first column a rest should remain, such rest can never be greater than the number of marks in the second column, because, if the rest was greater by one, a new mark would be made in the second column. The whole theory is based upon the fact that the rest in any column can never be greater than the number of marks in the next column. If the marks in the second third and all following columns, and if all the rests, are divided in the same way, that is, in the natural succession of the numbers 1, 2, 3; the numbers will appear in a very simple form, each consisting of a base aud a rest or correcting part, this rest or correcting part, however; is sometimes naught; thus:

One corrected by naught equals one, one corrected by itself equals two;

Two corrected by naught equals three, two corrected by one equals four, two corrected by itself equals five;

Three corrected by naught equals six, three corrected by one equals seven, three corrected by two equals eight, three corrected by itself equals nine;

Four corrected by naught equals ten, four corrected by one equals eleven, four corrected by two equals twelve, four corrected by three equals thirteen, four corrected by itself equals fourteen;

Five corrected by naught equals fifteen, five corrected by one equals sixteen, five corrected by two equals seventeen, five corrected by three equals eighteen, five corrected by four equals nineteen, fiive corrected by itself equals twenty.

The first twenty numbers are represented by the first twenty letters of the alphabet, excluding the letter *j*, so that *a* stands for 1, *b* stands for 2, &c., and *u* stands for 20. One of the last letters, for instance *x*, may stand for 0 (naught). After twenty, take the sixth letter, *f*, and say

6 corrected by 0 equals 21, equals *fx*,
6 " " 1 " 22, " *fa*,
6 " " 2 " 23, " *fb*,
6 " " 3 " 24, " *fc*,
6 " " 4 " 25, " *fd*,
6 " " 5 " 26, " *fe*,
6 " "itself" 27, " *ff*,

When the second part equals the first part, a stop is to be made and the first part is to be increased by one, hence we have:

7 corrected by 0 equals 28, equals *gx*,
7 " " 1 " 29, " *ga*,

and so forth until we arrive at 20 corrected by itself equals 230, equals *uu*. Now the first part *u*, is to be increased by one, which is *fx*, and we have:

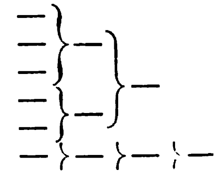
21 corrected by 0 equals 231, equals *fxx*,
21 " " 1 " 232, equals *fxa*.

In this way, all the numbers may be expressed, as the following table shows.

$x = 0$	$k = 10$	$u = 20$	$gb = 30$	$fxx = 231$
$a = 1$	$l = 11$	$fx = 21$	$gc = 31$	. .
$b = 2$	$m = 12$	$fa = 22$	$gd = 32$	. .
$c = 3$	$n = 13$	$fb = 23$	$ge = 33$	. .
$d = 4$	$o = 14$	$fc = 24$	$gf = 34$	. .
$e = 5$	$p = 15$	$fd = 25$	$gg = 35$	$uu\ uu = 26795$
$f = 6$	$q = 16$	$fe = 26$	. .	. .
$g = 7$	$r = 17$	$ff = 27$	. .	. .
$h = 8$	$s = 18$	$gx = 28$	. .	$uuuu\ uuuu$
$i = 9$	$t = 19$	$ga = 29$	$uu = 230$	$= 359026205.$

The general rule is: The second part is to be increased until it equals the first part, and then the first part is to be increased by one.

Every body may convince himself of the correctness of the proceeding by making six short horizontal dashes one over the other, and dividing the same according to the numbers 1, 2, 3, thus . . . .



then drawing 21 short horizontal dashes one over the other and dividing them according to the numbers 1, 2, 3, 4, 5, 6 and then interpolating the numbeas from 7 to 21.

SOLUTIONS OF PROBLEMS IN NUMBER FOUR.

Solutions of problems in No. 4, have been received as follows:

From Geo. L. Dake, 75 & 76; Prof. A. B. Evans, 75, 76, 77, 78, 79, 80, 81 & 82; E. S. Farrow, 75, 76 & 79; H. Heaton, 75, 76, 77, 78, 79, 80, 81 & 82; G. W. Hill, 81; Artemas Martin, 79 & 81; A. B. Nelson, 75, 79 & 81; L. Regan, 78; E. B. Seitz, 75, 77, 79 & 82; Walter Siverly, 75, 76, 77, 78, 79 & 82; Prof. C. M. Woodward, 76 & 77.

75. "A stick of timber of uniform density and size from end to end, has a weight of 600 pounds suspended at one end, by which it is balanced horizontally on a fulcrum 6 feet from the end where the weight is suspended. If the fulcrum be 5 feet from the same end the stick will be balanced with a weight of 800 pounds. Required the weight and length of the stick."

SOLUTION BY CADET E. S. FARROW, WEST POINT, N. Y.

Since the stick is uniform, its centre of gravity is at its middle point. Let l equal its length, and w its weight (A and B representing the two ends, G its centre and C' and C the positions of the 1st and 2nd named fulcrums respectively). Taking moments about C and C' we have

$$w\times CG + 0\times CA = 600\times CB,$$

also $$w\times C'G + 0\times C'A = 800\times C'B;$$

or $$w\times(\tfrac{1}{2}l - 6) = 3600, \quad \text{and} \quad w\times(\tfrac{1}{2}l - 5) = 4000;$$

whence $$l = 30, \quad \text{and} \quad w = 400.$$

76. "Within a given circle to draw three others tangent to it and touching each other externally, the sum of whose diameters shall be equal to that of the given circle."

77. "Within the circumference of a given circle, radius r, describe two equal circles, radii $\frac{2}{3}r$, which shall touch the first circle internally at opposite extremities of the diameter. Describe a fourth circle D, touching the first circle internally and each of the two equal circles externally. Describe a fifth circle E, a sixth circle F, &c., touching the first internally and one of the two equal circles and the circle D, E, F, &c., respectively, externally. Find an expression for the radii of the circles E, F, G, &c."

SOLUTION BY PROF. C. M. WOODWARD, ST. LOUIS, MO.

Preliminary Remarks. The properties of tangent circles used in the following solution are proved in an admirable article in the Mathematical Monthly, Vol. I. pp. 268 et seq., by Matthew Collins of Dublin. One of these properties may be stated thus:

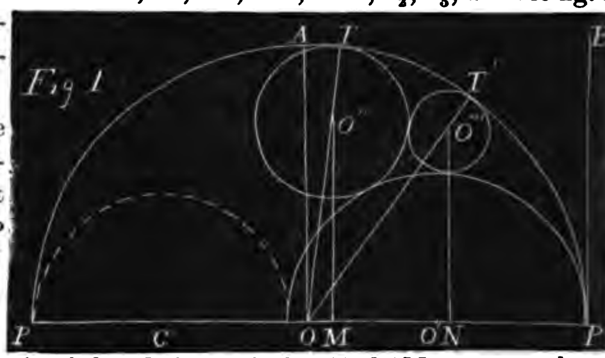
If a variable circle is tangent to two fixed circles, the distance of its centre from the radical axis of the fixed circles bears a constant ratio to its radius.

Another, a property of the "Arbelos," is older if not better known:

If O'' and O''' (Fig. 1.) *are any two circles tangent to each other and to O and O', and if the ratio*

$$\frac{O''M}{O''T} = n, \qquad \textit{then} \qquad \frac{O'''N}{O'''T'} = n + 2.$$

In the solution of problem 76, let the radius of the given circle $= 1$, and let the radii of the required circles, O', O'', O''', be r', r_2, r_3, and let fig. 1 be a general representation of the four circles.



By examining the special case of the dotted circle C, we see that

$$\frac{CP}{CP'} = \frac{MP}{O''T} = \frac{MP}{r_2} = \frac{NP}{r_3} = \frac{1 + r'}{1 - r'},$$

PB being the radical axis of O and O'; and, that if $O''M = nr_2$ we have $O'''N = (n + 2)r_3$. For brevity let

$$\frac{1 + r'}{1 - r'} = v, \text{ then} \begin{cases} O\ \ M = 1 - vr_2 \\ O''M = nr_2 \end{cases} \text{and} \begin{cases} O\ \ N = 1 - vr_3 \\ O'''N = (n + 2)r_3. \end{cases}$$

Hence

$$(1 - r_2)^2 = n^2 r_2^2 + (1 - vr_2)^2, \ldots\ldots \quad (1)$$

$$(1 - r_3)^2 = (n + 2)^2 r_3^2 + (1 - vr_3)^2, \ldots\ldots \quad (2)$$

$$1 = r' + r_2 + r_3. \qquad \ldots\ldots \quad (3)$$

Equations (1) and (2) give

$$r_2 = \frac{2(v - 1)}{n^2 + v^2 - 1}, \ldots (4) \text{ and } r_3 = \frac{2(v - 1)}{(n + 2)^2 + v^2 - 1} \ldots (5)$$

Since $1 - r' = \frac{2}{v + 1}$, the substitution of (4) and (5) in (3) gives

$$\frac{1}{n^2 + v^2 - 1} + \frac{1}{(n + 2)^2 + v^2 - 1} = \frac{1}{v^2 - 1} \ldots\ldots (6)$$

Solving for n we get $\qquad n = \pm v - 1.$

Negative values of n indicate either that the ordinate of the centre of r_2 is negative, (i. e. O'' is in the other horn of the Arbelos), or that r_2 is itself negative, O'' being external to O. Hence, omitting the negative value of v, we have from (4) and (5)

$$r' = \frac{v-1}{v+1}, \quad r_2 = \frac{1}{v} = \frac{1-r'}{1+r'}, \quad r_3 = \frac{v-1}{v+1} \cdot \frac{1}{v} = r'r_2.$$

Moreover we have $OM = 1 - vr_2 = 0,$

$$ON = 1 - vr_3 = 1 - r' = OO',$$

$$O''M = nr_2 = \frac{v-1}{v} = (n+2)r_3 = O'''N.$$

These results show, when taken in connection with the remark as to the negative value of v;—

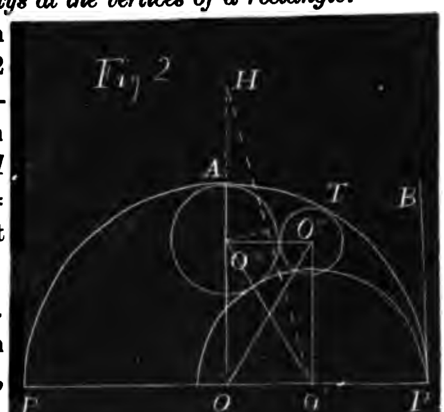
1°. That r' may be taken equal to any number, integral or fractional, positive or negative; as $a \div b$.

2°. That r_2 is given by a fraction whose numerator equals the denominator of r' *minus* its numerator; and whose denominator is the sum of the denominator and numerator of r'; as $(b-a) \div (b+a)$.

3°. That r_3 is the product of r' and r_2; as

$$\frac{a}{b} \cdot \frac{b-a}{b+a}.$$

4°. That *the four centres are always at the vertices of a rectangle.*



Sets of values of r', r_2, and r_3, can readily be given *ad infinitum.* Fig. 2 shows the circles in their correct position. The geometrical construction of any required set is indicated, AH being equal to $O'P$. Since $OO' = O''O''' = r_2 + r_3$ it is obvious that

$$r' + r_2 + r_3 = 1.$$

Cor. I. If $r' = \frac{2}{3}$ we have a ready solution of problem 77. In this case $v = 5$, $n = 4$ and, Fig. 3,

$$r_2 = \text{radius of } D = \tfrac{1}{5}$$
$$r_3 = \text{radius of } E = \tfrac{2}{15},$$
$$r_4 = \text{“ “ } F = \tfrac{2}{22} = \tfrac{1}{11},$$

. . . .

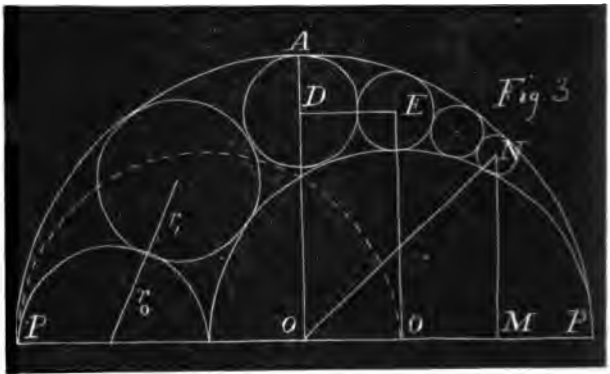
and obviously $$r_m = \frac{2(v-1)}{[n+2(m-2)]^2 + v - 1} = \frac{2}{m^2+6}; \quad \ldots\ldots \quad (7)$$

hence $r_0 = \frac{1}{3}$, $r_1 = \frac{2}{7}$, $r_2 = \frac{1}{5}$, $r_3 = \frac{2}{15}$, $r_4 = \frac{1}{11}$, $r_5 = \frac{2}{31}$, $r_6 = \frac{1}{21}$ &c.

Cor. II. Since $v = (1+r') \div (1-r')$ and $n = 2r' \div (1-r')$, (7) gives

$$r_m = \frac{r'(1-r')}{[(3-m)r' + m - 2]^2 + r'}, \quad \ldots\ldots\ldots \quad (8)$$

which gives in terms of r' (which may be any number) the radius of any circle in a series, the centre of r_2 being on the radius OA ⊥ to PP'.



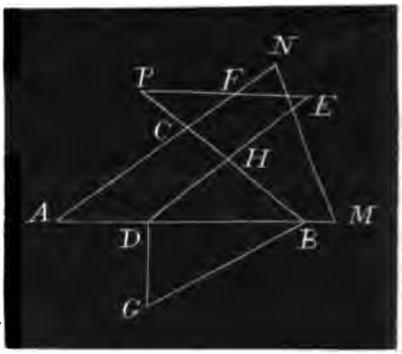
Cor. III. If $n = 2r' \div (1 - r') = v - 1$ is an even number, repeated subtractions of 2 must reduce it to zero, so that the largest circle of the series has its centre on PP', and the series may be called *symmetrical.* This is the case when v is an odd integer, as in Fig. 3.

---

78. "From a given point without a triangle to draw a line which bisects the area of the triangle."

SOLUTION BY PROF. A. B. EVANS, LOCKPORT, N. Y.

Let NAM be the given triangle and P the given point. Draw PE parallel to AM and construct the parallelogram $AFED$ equal to one half the area of the triangle NAM. Draw DG perpendicular to AM and equal to PF, and GB equal to PE.



From the similar triangles PHE, PCF, BDH, since $PE^2 - PF^2 = DB^2$

$PHE - PCF = BDH$,

$\therefore \quad ACB = AFED = \frac{1}{2}NAM.$

Note. In Chauvenet's Geometry, exercise 249, p. 328, we find the following question which is more general than No. 78, and which may be solved by making the parallelogram $AFED$ equal to the part to be cut off from the angle NAM by a line through P:

"Through a given point, draw a straight line which shall form with two given intersecting straight lines a triangle of a given area."

---

79. "Prove that the formula $2^{n-1}(2^n - 1)$ represents a perfect number when $2^n - 1$ is prime."

SOLUTION BY E. B. SEITZ, GREENVILLE, OHIO.

A number which is equal to the sum of all its divisors 1 inclusive, is called a perfect number.

When $2^n - 1$ is prime the sum of all the divisors of $2^{n-1}(2^n - 1)$ including the number itself

$$= (1 + 2 + 2^2 + \ldots\ldots + 2^{n-1})(1 + 2^n - 1) = 2^n(2^n - 1),$$

which is twice the number. Therefore $2^{n-1}(2^n - 1)$ is a perfect number when $2^n - 1$ is prime.

---

80. "The relation

$$\underline{|n} = 1 + A_1\frac{n}{\underline{|1}} + A_2\frac{n(n-1)}{\underline{|2}} + A_3\frac{n(n-1)(n-2)}{\underline{|3}} + \ldots + A_n\frac{\underline{|n}}{\underline{|n}}$$

is true for all positive integral values of n; show that

$$\frac{A_r}{\underline{|r}} = 1 - \frac{1}{\underline{|1}} + \frac{1}{\underline{|2}} - \frac{1}{\underline{|3}} \ldots\ldots + \frac{(-1)^r}{\underline{|r}},$$

where r is an integer less than n."

[There was a slight misprint in the announcement of this question. "Ar" should have been A_r.]

SOLUTION BY H. HEATON, DES MOINES, IOWA.

Put $A_r = x^r = [(x+1) - 1]^r = (x+1)_r$

$$- \frac{r}{\underline{|1}}\left(x+1\right)^{r-1} + \frac{r(r-1)}{\underline{|2}}\left(x+1\right)^{r-2} \ldots + (-1)^r$$

Then $A_r = \left(A_r + \frac{r}{\underline{|1}}A_{r-1} + \frac{r(r-1)}{\underline{|2}}A_{r-2} \ldots + 1\right) - \frac{r}{\underline{|1}}\left(A_{r-1} + \frac{r-1}{\underline{|1}}\right.$

$\left.\times A_{r-2} \ldots + 1\right) + \frac{r(r-1)}{\underline{|2}}\left(A_{r-2} + \frac{r-2}{\underline{|1}}A_{r-3} + \ldots + 1\right) \ldots \left(-1\right)^r.$

But from the conditions of the problem it is easily seen that the quantity in the first parenthesis $= \underline{|r},$ that in the second $= \underline{|r-1}$ that in the third $= \underline{|r-2}$ &c. Hence

$$A_r = \underline{|r} - \frac{\underline{|r}}{\underline{|1}} + \frac{\underline{|r}}{\underline{|2}} - \frac{\underline{|r}}{\underline{|3}} \ldots + \frac{(-)^r\underline{|r}}{\underline{|r}},$$

$$\therefore \quad \frac{A_r}{\underline{|r}} = 1 - \frac{1}{\underline{|1}} + \frac{1}{\underline{|2}} - \frac{1}{\underline{|3}} \ldots + \frac{(-1)^r}{\underline{|r}}.$$

---

81. "Prove that, *identically*,

$$\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} \ldots + \frac{1}{2n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots - \frac{1}{2n}."$$

SOLUTION BY ARTEMAS MARTIN, ERIE, PA.

Let $\quad P = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \ldots + \frac{1}{2n-1} \ldots\ldots\ldots\ldots \quad (1)$

and $\quad Q = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \ldots + \frac{1}{2n} \ldots\ldots\ldots\ldots \quad (2)$

By subtraction, $P - Q = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots - \frac{1}{2n} \ldots\ldots \quad (3)$

By addition, $\quad P + Q = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{2n} \ldots\ldots \quad (4)$

Subtracting twice (2) from (4),

$$P - Q = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \ldots + \frac{1}{2n}$$

---

82. "If the parabolic orbits of two comets intersect the circular orbit of the earth in the same two points, then if t_1 and t_2 be the times in which the comets move from one point to the other, $(t_1 + t_2)^{2/3} + (t_1 - t_2)^{2/3}$

$$= \left(\frac{4}{3\pi}\right)^{2/3}:$$

a year being the unit of time."

SOLUTION BY WALTER SIVERLY, OIL CITY, PA.

Let c represent the length of the chord of the earth's orbit joining the two points. It is shown in works on Central Forces that,

$$t_1 = \frac{1}{12\pi}\Big(2 + c\Big)^{\frac{3}{2}} + \Big(2 - c\Big)^{\frac{3}{2}},$$

$$t_2 = \frac{1}{12\pi}\Big(2 + c\Big)^{\frac{3}{2}} - \Big(2 - c\Big)^{\frac{3}{2}}.$$

Adding and subtracting,

$$t_1 + t_2 = \frac{1}{6\pi}\Big(2 + c\Big)^{\frac{3}{2}}, \qquad t_1 - t_2 = \frac{1}{6\pi}\Big(2 - c\Big)^{\frac{3}{2}}.$$

Whence

$$\Big(t_1 + t_2\Big)^{\frac{2}{3}} = \Big(2 + c\Big)\left(\frac{1}{6\pi}\right)^{\frac{2}{3}},$$

$$\Big(t_1 - t_2\Big)^{\frac{2}{3}} = \Big(2 - c\Big)\left(\frac{1}{6\pi}\right)^{\frac{2}{3}}.$$

Eliminating c,

$$\Big(t_1 + t_2\Big)^{\frac{2}{3}} + \Big(t_1 - t_2\Big)^{\frac{2}{3}} = \left(\frac{4}{3\pi}\right)^{\frac{2}{3}}.$$

PARADOX. BY G. SHAW, KEMBLE, ONT., CANADA.—Suppose $a = b$. Multiply both sides of this equality by b, and we have $ab = b^2$. Subtract a^2 from each side of this equation and we have $ab - a^2 = b^2 - a^2 \ldots$ (1) By factoring (1) we have $(b - a)a = (b - a)(b + a)$. Divide this eqn. by $b - a$ and we have $a = b + a$ or, because $b = a$,

$$a = a + a = 2a.$$

Dividing by a we have $1 = 2$.

[The fallacy in the above obviously consists in considering a and b as *like* and *unlike* at the same time. For, in the first member of (1) because $a = b$, we have a zero factor, $b - a = 0$, and in the second member we have the same zero factor, $b - a$. Now when a and b are unlike the only two factors of $b^2 - a^2$ are $a - b$ and $a + b$, neither of which can be zero, but when $b = a$ then $b^2 - a^2 = a^2 - a^2$, the only two factors of which are $a - a$ and a; the first of these two factors, $a - a$, being zero, the other factor a, may be repeated as often as we please without changing the value of the product.—Ed]

PROBLEMS.

83. BY GEO. L. DAKE, CLEVELAND, OHIO.—A point is given within two lines which form a given angle with one another. Required the shortest line which can be drawn through this point, terminated by the given lines.

84. BY PHILLIP HOGLAN, NEWCOMERSTOWN, OHIO.—The centres of two spheres whose radii are 12 ft. and 5 ft., respectively, are at opposite extremities of the diameter of a circle of 13 ft. radius. Find a point in the cirumference of this circle from which the greatest portion of spherical surface is visible.

85. BY PROF. JAMES G. CLARK, LIBERTY, MO.—In a quadrilateral there are given, the length and position of the lower base, the lengths of the two sides, the length of the upper base and the position of a point through which it passes: required to construct the quadrilateral.

86. BY PROF. J. S. HAYES, HODGENVILLE, KY.—Prove that the attraction of a sphere of uniform density upon an external point is the same as if all the matter of the sphere were concentrated at its centre.

87. BY G. M. DAY, LOCKPORT, N. Y.—There are n tickets in a bag numbered $1, 2, 3 \ldots n$. A man draws three tickets together at random and is to receive a number of shillings equal to the product of the numbers he draws. Find the value of his expectation.

88. By Prof. H. T. J. Ludwick, Salisbury, N. C.— An ellipse revolves about its latus rectum; show that the volumes of the solids generated by the larger and smaller segments are respectively equal to

$$\frac{4\pi a^3}{3}\left(1-e^2\right)\left[\frac{2+e^2}{2}+\frac{3e}{(1-e^2)^{\frac{1}{2}}}\tan^{-1}\left(\frac{1+e}{1-e}\right)^{\frac{1}{2}}\right], \quad \text{and}$$

$$\frac{4\pi a^3}{3}\left(1-e^2\right)\left[\frac{2+e^2}{2}-\frac{3e}{(1-e^2)^{\frac{1}{2}}}\tan^{-1}\left(\frac{1-e}{1+e}\right)^{\frac{1}{2}}\right].$$

89. By Artemas Martin, Erie, Pa.—A sphere, radius r, *rolls* down the surface of another sphere of the same material, radius R, placed on a horizontal plane. The surfaces of both spheres and plane are rough enough to secure perfect rolling. Determine the motion of the spheres, the point of separation and the equation of the curve described by the center of the upper sphere.

90. By R. J. Adcock, Monmouth, Ill.—Let an oblate ellipsoid of revolution of homogeneous density rotate about one of its greatest diameters. What must be the ratio of its axes, that a column of liquid along the greatest diameter at right angles to the axis of rotation may just balance one along the shortest diameter? $a^2 \div \delta k_1$ being the same as in the case of the earth, where $a=$ angular velocity of rotation, $k_1=$ attraction of a spherical unit of mass for another at the distance unity between centres, $\delta=$ mean density.

91. By Prof. W. W. Johnson, Annapolis, Md.—Let a sphere, rotating with the angular velocity w, be provided with pivots at the extremities of a diameter inclined to the axis at the angle a. If these pivots be suddenly caught in fixed sockets, the sphere will rotate about the new axis with the rate $w\cos a$. If the pivots be caught by a ring which is itself free to rotate about an axis perpendicular to the new axis of the sphere and passing through its centre, the rate of rotation about this axis will be $w\sin a$. The original rotation and these component rotations represent kinetic energies which are proportional to w^2, $w^2\cos^2 a$ and $w^2\sin^2 a$: hence there is no loss of energy and no shock. That is, every particle will, when the pivots are caught, undergo no sudden change in velocity or direction. Prove the truth of this by spherical trigonometry.

Note. If solutions of the problems proposed in any No. are received by the 10th of the next succeeding month, they will in general be either published in the first succeeding No., or noticed at the head of "*Solutions of Problems*"; but if received after the 10th of such month, and if a solution of the problem, or problems, is published in the next No., no notice in general is given of such solution.—Ed.

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ON THE DEVELOPMENT OF THE PERTURBATIVE FUNCTION IN PERIODIC SERIES.

BY G. W. HILL, NYACK TURNPIKE, N. Y.

1. THERE are two modes of developing this function. In one, the numerical values of the elements involved are employed from the outset, and the results obtained belong only to the special case treated. This mode has been, almost exclusively, followed by Hansen, and is, perhaps, to be recommended when numerical results are chiefly desired. In the other, all the elements are left indeterminate, and thus is obtained a literal development possessing as much generality as possible. Certain investigations, arising from Jacobi's treatment of dynamical equations and Delaunay's method in the lunar theory, have invested the latter mode of development with additional interest, and with it we shall be exclusively engaged in this article.

In Liouville's Journal for 1860, M. Puiseux has given us two memoirs on this subject, in which appears the general term of this function, but his formulæ seem susceptible of modifications which would render them much simpler. More recently, in the volume of the same journal for 1873, M. Bourget has presented the development in a more concise form by employing the Besselian functions, but as he discards the use of the functions $b_s^{(i)}$, his formulæ, on this account are more complex. It is hoped, that, even if the expressions, given hereafter, are deemed too cumbrous for practical use, they may still possess some interest from a theoretical point of view.

2. It is known that if we have a function S of a variable ζ, which is never infinite, and such that the relation

$$\text{function}\ (\zeta + 2i\pi) = \text{function}\ (\zeta)$$

is satisfied for all integral values of i both positive and negative, it can be developed in a series of the form

$$\Sigma_i\,.\,(K_i^{(c)} \cos i\zeta + K_i^{(s)} \sin i\zeta),$$

in which i denotes a positive integer; and that, in the cases where this series is infinite, it is convergent.

In general the handling of periodic series is easier if we introduce imaginary exponentials in the place of the circular functions. Thus, ϵ denoting the base of natural logarithms, we shall put $z = \epsilon^{\zeta\sqrt{-1}}$, whence

$$2\cos\zeta = z + z^{-1}, \qquad 2\cos i\zeta = z^i + z^{-i},$$

$$2\sqrt{(-1)}\sin\zeta = z - z^{-1}, \qquad 2\sqrt{(-1)}\sin i\zeta = z^i - z^{-i},$$

$$z = \cos\zeta + \sqrt{(-1)}\sin\zeta, \qquad z^i = \cos i\zeta + \sqrt{(-1)}\sin i\zeta.$$

The above theorem then comes to the same thing as to say that S is developable in a series of the form

$$\Sigma_i \,.\, C_i z^i,$$

where the summation is extended to negative as well as positive values of i. The coefficients K are given in terms of the coefficients C by the equations

$$K_i^{(c)} = C_i + C_{-i},$$

$$K_i^{(s)} = (C_i - C_{-i})\sqrt{-1},$$

except the case where $i = 0$, when $K_0^{(c)} = C_0$. It will be seen that when S is real, C_i is a complex number $a + b\sqrt{-1}$, and C_{-i}, its conjugate $a - b\sqrt{-1}$, which renders the coefficients K real, as they should be.

The integral

$$\int z^i d\zeta = \int\left(\cos i\zeta + \sqrt{(-1)}\sin i\zeta\right)d\zeta,$$

taken between the limits 0 and 2π, vanishes in all cases except when $i=0$, when its value is 2π. Hence any function, capable of expansion in a series of positive and negative integral powers of z, integrated with respect to ζ between these limits, gives, as the result, 2π times the coefficient of z^0 in its expansion. And as the coefficient of z^0 in the function Sz^{-i} is evidently C_i, we have

$$C_i = \frac{1}{2\pi}\int_0^{2\pi} Sz^{-i}\, d\zeta.$$

This equation holds for all values of i, negative as well as positive, zero included.

3. Let us now suppose that ζ denotes the mean anomaly of a planet, and let u be the eccentric anomaly, connected with the former by the equation, e being the eccentricity,

$$u - e\sin u = \zeta.$$

In like manner as for ζ, we will introduce the imaginary exponential $s = \epsilon^{u\sqrt{-1}}$, thus, as the last equation can be written

$$\varepsilon^{(u - e\sin u)\sqrt{-1}} = \varepsilon^{\zeta\sqrt{-1}},$$

by the introduction of the variables s and z, this becomes

$$s\varepsilon^{-\frac{e}{2}\left(s-\frac{1}{s}\right)} = z,$$

which is the transcendental equation connecting s and z. We have

$$d\zeta = (1 - e\cos u)du = \left[1 - \frac{e}{2}\left(s + \frac{1}{s}\right)\right]du.$$

Substituting these values in the equation giving the value of C_i, and noticing that, as ζ and u both take the values 0 and 2π together, the limits of integration, when u is the independent variable, are the same as for ζ, we get

$$C_i = \frac{1}{2\pi}\int_0^{2\pi} Ss^{-i}\varepsilon^{\frac{ie}{2}\left(s-\frac{1}{s}\right)}\left[1 - \frac{e}{2}\left(s + \frac{1}{s}\right)\right]du$$

But, from what precedes, we conclude that the coefficient of s^i in the expansion of any function W, according to positive and negative powers of s, is

$$\frac{1}{2\pi}\int_0^{2\pi} Ws^{-i}du.$$

Thus, from the foregoing expression for C_i, we derive the following proposition:—

i *being a positive or negative integer or zero, the coefficient of* z^i, *in the development of* S, *according to the powers of* z, *is equal to that of* s^i *in the development of*

$$S\varepsilon^{\frac{ie}{2}\left(s-\frac{1}{s}\right)}\left[1 - \frac{e}{2}\left(s + \frac{1}{s}\right)\right],$$

according to the powers of s.

As most of the functions S, which are presented by astronomy for development in powers of z, are quite readily expanded in powers of s, this theorem is of much use. Another form can be given to it. For we have, integrating by parts

$$\int Sz^{-i}d\zeta = -\sqrt{(-1)}\int Sz^{-(i+1)}dz$$

$$= \frac{\sqrt{-1}}{i}Sz^{-i} - \frac{\sqrt{-1}}{i}\int\frac{dS}{dz}z^{-i}dz.$$

Taking the integrals between the limits $\zeta = 0$ and $\zeta = 2\pi$, we get

$$C_i = -\frac{\sqrt{-1}}{2i\pi}\int\frac{dS}{ds}z^{-i}ds$$

$$= \frac{1}{2i\pi}\int_0^{2\pi}\frac{dS}{ds}\varepsilon^{\frac{ie}{2}\left(s-\frac{1}{s}\right)}s^{-(i-1)}du.$$

Whence we conclude this proposition: —

The coefficient of z^i *in the development of* S *according to the powers of* z *is equal to that of* s^{i-1} *in the development of*

$$\frac{1}{i}\cdot\frac{dS}{ds}\,\varepsilon^{\frac{ie}{2}\left(s-\frac{1}{s}\right)}$$

according to the powers of s.

This theorem however is not applicable when $i = 0$.

4. We shall often have occasion for the expansion of the function

$$\varepsilon^{\frac{ie}{2}\left(s-\frac{1}{s}\right)}$$

in powers of s; let us, for simplicity, put $\lambda = \frac{ie}{2}$, and

$$\varepsilon^{\lambda\left(s-\frac{1}{s}\right)} = \varepsilon^{\lambda s}\,.\,\varepsilon^{-\frac{\lambda}{s}} = \Sigma_i\,.\,J^{(i)}_{\lambda}\,s^i.$$

We have

$$\varepsilon^{\lambda s}\,.\,\varepsilon^{-\frac{\lambda}{s}} = \left[1+\frac{\lambda s}{1}+\frac{\lambda^2 s^2}{1\,.\,2}+\frac{\lambda^3 s^3}{1\,.\,2\,.\,3}+\dots\right]$$
$$\times\left[1-\frac{\lambda}{s}+\frac{1}{1\,.\,2}\frac{\lambda^2}{s^2}-\frac{1}{1\,.\,2\,.\,3}\frac{\lambda^3}{s^3}+\dots\right],$$

whence we conclude that

$$J^{(i)}_{\lambda} = \frac{\lambda^i}{1\,.\,2\,.\,.\,.\,i}\left[1-\frac{\lambda^2}{1.(i+1)}+\frac{\lambda^4}{1.2(i+1)(i+2)}-\dots\right].$$

This series is not applicable when i is negative; but if, in the function $\varepsilon^{\lambda s}\,.\,\varepsilon^{-\frac{\lambda}{s}}$, we substitute $\frac{1}{s}$ for s, and change the sign of λ, the function remains unchanged, hence

$$\Sigma_i\,.\,J^{(i)}_{\lambda}\,s^i = \Sigma_i\,.\,J^{(i)}_{-\lambda}\,s^{-i},$$

and consequently

$$J^{(-i)}_{\lambda} = J^{(i)}_{-\lambda} = (-1)^i J^{(i)}_{\lambda},$$

by which the values of these functions for negative values of i can be derived from those in which i is positive. These functions are known as the Besselian. By putting

$$T_i = 1-\frac{\lambda^2}{1.(i+1)}+\frac{\lambda^4}{1.2(i+1)(i+2)}-\dots$$

one will have no difficulty in deducing the equation

$$T_{i-1} = T_i - \frac{\lambda^2}{i(i+1)}\,T_{i+1}.$$

5. We come now to the more complex function S of two variables ζ and ζ'; it is known that when this is never infinite and is such that

$$\text{function } (\zeta + 2i\pi, \zeta' + 2i'\pi) = \text{function } (\zeta, \zeta')$$

it can be developed in a series of the form

$$\Sigma_{i,\,i'} \left[K^{(c)}_{i,\,i'} \cos (i\zeta + i'\zeta') + K^{(s)}_{i,\,i'} \sin (i\zeta + i'\zeta')\right],$$

where to one of the quantities i and i', we need assign only positive integral values, but to the other both positive and negative values. If we adopt another imaginary exponential $z' = \varepsilon^{\zeta'\sqrt{-1}}$, this is the same as saying that

$$S = \Sigma_{i,\,i'} \,.\, C_{i,\,i'} z^i z'^{i'},$$

where the summation is extended to all integral values positive and negative for i and i'. Since we have

$$\begin{aligned} z^i z'^{i'} &= (\cos i\zeta + \sqrt{(-1)} \sin i\zeta)(\cos i'\zeta' + \sqrt{(-1)} \sin i'\zeta') \\ &= \cos (i\zeta + i'\zeta') + \sqrt{(-1)} \sin (i\zeta + i'\zeta'), \end{aligned}$$

the relations, which connect the coefficients K with the coefficients C, are

$$\begin{aligned} K^{(c)}_{i,\,i'} &= C_{i,\,i'} + C_{-i,-i'}, \\ K^{(s)}_{i,\,i'} &= (C_{i,\,i'} - C_{-i,-i'})\sqrt{-1}, \end{aligned}$$

unless i and i' are both zero, when

$$K^{(c)}_{0,\,0} = C_{0,\,0} \,.$$

A course of reasoning, similar to that in the case of one variable, establishes that

$$C_{i,\,i'} = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} S z^{-i} z'^{-i'} d\zeta\, d\zeta',$$

which holds for all integral values of i and i', positive, negative and zero.

6. Supposing that ζ' denotes the mean anomaly of a second planet, whose eccentricity and eccentric anomaly are respectively e' and u', we have

$$u' - e' \sin u' = \zeta',$$

and by the adoption of the imaginary exponential $s' = \varepsilon^{u'\sqrt{-1}}$, this is transformed into

$$s' \varepsilon^{-\frac{e'}{2}(s' - \frac{1}{s'})} = z'.$$

It is not difficult to see that we have the following theorem:—

The coefficient of $z^i z'^{i'}$ *in the development of* S, *according to the powers of* z *and* z', *is equal to that of* $s^i s'^{i'}$ *in the development of*

$$S \varepsilon^{\frac{ie}{2}(s - \frac{1}{s})}\left[1 - \frac{e}{2}\left(s + \frac{1}{s}\right)\right] \,.\, \varepsilon^{\frac{i'e'}{2}(s' - \frac{1}{s'})}\left[1 - \frac{e'}{2}\left(s' + \frac{1}{s'}\right)\right],$$

according to the powers of s *and* s'.

7. After these preliminaries relative to the general development of functions in periodic series, we come to the matter more immediately engaging our attention. The perturbative function for the action of a planet,

whose mass is m', on another, whose mass is m, is usually written

$$R = m'\left[\frac{1}{\varDelta} - \frac{r\cos\psi}{r'^2}\right],$$

and that for the action of m on m'

$$R_1 = m\left[\frac{1}{\varDelta} - \frac{r'\cos\psi}{r^2}\right],$$

where $\varDelta$ denotes their mutual distance, ψ their angular distance as seen from the sun, and r and r' their radii vectores. The problem proposed is then to develop these two functions in series whose general term is of the form $C_{i,i'}z^i z'^{i'}$. To this end it seems better to discuss the two portions of the general perturbative function, $\frac{1}{\varDelta}$ and $-\frac{r\cos\psi}{r'^2}$, separately, and not, as most investigators, attempt, by a particular notation, to combine, in a whole, these two parts. Thus, in developing $\frac{1}{\varDelta}$, we shall have the term common to both functions, and may suppose that r' denotes the radius vector which belongs to the planet more distant from the sun. But, in treating the second part, we shall suppose that r' belongs to the disturbing planet. The following equations are well known,

$$\begin{aligned}
\varDelta^2 &= r'^2 - 2rr'\cos\psi + r^2,\\
\cos\psi &= \cos(v+\Pi)\cos(v'+\Pi') + \cos I\sin(v+\Pi)\sin(v'+\Pi')\\
&= \cos(v-v'+\Pi-\Pi') - 2\sin^2\tfrac{1}{2}I\sin(v+\Pi)\sin(v'+\Pi'),
\end{aligned}$$

where v and v' are the true anomalies, and Π and Π' are the angular distances of the perihelia from either point of intersection of the planes of the orbits, and I is their mutual inclination.

8. Attending then, in the first place, to the development of $\frac{1}{\varDelta}$, we have to notice what are the conditions under which this quantity can be developed in powers of z and z'. In the case of two elliptic orbits, the only one we shall consider here, it is plain that $\frac{1}{\varDelta}$ is always finite and continuous provided the orbits have no point in common. Here we must make two cases according as the value of $\sin I$ is not or is zero. In the first case it is evident that the orbits can meet only on the line of intersection of their planes. Hence, p and p' denoting their semi-parameters, there will be two, one or no points in common, according as two, one or none of the equations,

$$\begin{aligned}
p'(1+e'\cos\Pi')^{-1} &= p(1+e\cos\Pi)^{-1},\\
p'(1-e'\cos\Pi')^{-1} &= p(1-e\cos\Pi)^{-1},
\end{aligned}$$

are satisfied. In the second case, where the orbits lie in the same plane,

there will be two intersections or none, according as the equation

$$p'[1 + e'\cos(\lambda - \omega')]^{-1} = p[1 + e\cos(\lambda - \omega)]^{-1},$$

λ being the unknown quantity and ω and ω' the longitudes of the perihelia, admits real or imaginary roots. If we put

$$pe'\cos\omega' - p'e\cos\omega = A\cos a,$$
$$pe'\sin\omega' - p'e\sin\omega = A\sin a,$$

this equation takes the form

$$A\cos(\lambda - a) = p' - p.$$

The roots of this are imaginary when

$$(p' - p)^2 > p^2e'^2 - 2pp'ee'\cos(\omega - \omega') + p'^2e^2.$$

9. If we put

$$P = r'^2 - 2rr'\cos(v - v' + \Pi - \Pi') + r^2,$$
$$Q = 4\sin^2\tfrac{1}{2}I \,.\, r\sin(v + \Pi) \,.\, r'\sin(v' + \Pi'),$$

we have

$$\Delta^2 = P + Q,$$
$$\frac{1}{\Delta} = [P + Q]^{-\frac{1}{2}}$$
$$= P^{-\frac{1}{2}} - \frac{1}{2}P^{-\frac{3}{2}}Q + \frac{1}{2}\cdot\frac{3}{4}P^{-\frac{5}{2}}Q^2 - \ldots\ldots,$$

a series we shall denote thus

$$\frac{1}{\Delta} = \sum_{k=0}^{k=\infty}(-1)^k\,\frac{1.3\ldots(2k-1)}{2.4\ldots\,2k}\,P^{-\frac{2k+1}{2}}Q^k.$$

10. In order that this development of $\frac{1}{\Delta}$ in a series of ascending powers of Q, or, if one likes, of $\sin^2\frac{1}{2}I$, may be legitimate, it is necessary that the elements of the orbits should be such that the numerical value of $\frac{Q}{P}$ should be always less than unity. P is the square of the distance of the two planets after the plane of the orbit of one has been brought into coincidence with the plane of the other by revolving it about the line of intersection of the two planes. Taking then a system of rectangular axes passing through the centre of the sun, and directing the axis of x along the line of intersection, it is plain the equations of the orbits may be written

$$\sqrt{(x^2 + y^2)} + ax + \beta y = p,$$
$$\sqrt{(x'^2 + y'^2)} + a'x' + \beta'y' = p',$$

a, β, a', β' being constants. And the variables x, y, x', y' satisfying these

equations, the question depends on the finding of the values of them which render the expression

$$D = \frac{yy'}{(x-x')^2 + (y-y')^2}$$

a maximum or a minimum. According to the known theory of maxima and minima, the equations, which, in combination with the equations of the orbits, give these values, are

$$-2D(x-x') + \mu\left[\frac{x}{\sqrt{(x^2+y^2)}} + a\right] = 0,$$

$$2D(x-x') + \mu'\left[\frac{x'}{\sqrt{(x'^2+y'^2)}} + a'\right] = 0,$$

$$y' - 2D(y-y') + \mu\left[\frac{y}{\sqrt{(x^2+y^2)}} + \beta\right] = 0,$$

$$y + 2D(y-y') + \mu'\left[\frac{y'}{\sqrt{(x'^2+y'^2)}} + \beta'\right] = 0,$$

where μ and μ' are the multipliers of the partial derivatives of the two equations of condition. A complete investigation of this question would be conducted in the following manner. Eliminate from the seven equations last given the six quantities x, y, x', y', μ, μ'; the result will be an algebraical equation determining the unknown D. Having derived the Sturmian functions of this, one will ascertain by the substitution of the values $D = \frac{1}{4\sin^2\frac{1}{2}I}$, $D = +\infty$, and again of $D = -\frac{1}{4\sin^2\frac{1}{2}I}$, $D = -\infty$, whether any roots lie between these limits; if none, $\frac{1}{\Delta}$ can be expanded in a series of ascending powers of $\sin^2\frac{1}{2}I$, in the contrary case not. In this way we shall arrive at the condition or conditions necessary and sufficient for the legitimacy of this expansion.

11. This procedure would doubtless lead to very complicated formulæ, hence we are obliged to pass over it. However equations can be readily got, which, by a tentative process, afford the maxima and minima values of D. Multiply the four equations last given respectively by x, x', y, y' and add the resulting equations, having regard to the equations of the orbits and the value of D, we thus arrive at the simple relation

$$p\mu + p'\mu' = 0.$$

Putting, for simplicity, $x = r\cos\theta$, $x' = r'\cos\theta'$, the addition of the first and second of the same group of four equations gives

$$\mu(\cos\theta + a) + \mu'(\cos\theta' + a') = 0.$$

By combining this with the preceding is obtained

$$\frac{\cos\theta' + a'}{p'} = \frac{\cos\theta + a}{p}.$$

Again the addition of the same equations, multiplied severally by x, $-x'$, y, $-y'$, gives the equation

$$2D(r'^2 - r^2) = p'\mu' - p\mu.$$

Dividing the left member of this by $2D(x'-x)$, and the terms of the right member by its equivalents derived from the first and second equations, we get

$$\frac{r'^2 - r^2}{x' - x} = \frac{p'}{\cos\theta' + a'} + \frac{p}{\cos\theta + a'},$$

or

$$\frac{r'\cos\theta' - r\cos\theta}{r'^2 - r^2} = \frac{\cos\theta + a}{2p}.$$

This and the equation

$$\frac{\cos\theta' + a'}{p'} = \frac{\cos\theta + a}{p}$$

determine the values of the variables θ and θ' which render D a maximum or minimum. When the orbits are nearly circular these values are in the neighborhood of $\frac{1}{2}\pi$ or $\frac{3}{2}\pi$. When both orbits are circles the solution is very simple, and we find that in order the development may be legitimate, we must have

$$\sin\frac{I}{2} < \frac{a' - a}{2\sqrt{(aa')}},$$

a and a' being the mean distances of the planets from the sun.

12. Assuming that this development is legitimate, we have to develop $P^{-\frac{2k+1}{2}}Q^k$ in terms of s and s'. We have

$$r\cos v = a\,(\cos u - e) = \frac{a}{2}\left(s + \frac{1}{s} - 2e\right),$$

$$r\sin v = a\sqrt{(1-e^2)}\sin u = \frac{a}{2\sqrt{-1}}\sqrt{(1-e^2)}\left(s - \frac{1}{s}\right),$$

whence

$$r\cos v + r\sin v\,.\sqrt{-1} = r\varepsilon^{v\sqrt{-1}}$$
$$= a\left[\frac{1+\sqrt{(1-e^2)}}{2}s + \frac{1-\sqrt{(1-e^2)}}{2}\frac{1}{s} - e\right],$$

and by putting

$$\frac{1+\sqrt{(1-e^2)}}{2} = \eta, \quad \frac{e}{1+\sqrt{(1-e^2)}} = \omega,$$

we get

$$r\varepsilon^{v\sqrt{-1}} = a\eta s\left(1 - \frac{\omega}{s}\right)^2.$$

And the value of $r\varepsilon^{-v\sqrt{-1}}$ is evidently obtained by substituting in this $\frac{1}{s}$ for s, hence
$$r\varepsilon^{-v\sqrt{-1}} = a\eta\frac{1}{s}(1-\omega s)^2.$$

From these two equations may be derived
$$r = a\eta(1-\omega s)\left(1-\frac{\omega}{s}\right),$$
$$\varepsilon^{v\sqrt{-1}} = \frac{s-\omega}{1-\omega s}.$$

Writing γ for $\Pi - \Pi'$, we have
$$(r'^{-2}P)^{-\frac{2k+1}{2}} = \left[1-2\frac{r}{r'}\cos(v-v'+\gamma)+\frac{r^2}{r'^2}\right]^{-\frac{2k+1}{2}}.$$

The right member of this is developable in a series of integral powers of the exponential $\varepsilon^{(v-v'+\gamma)\sqrt{-1}}$ when $\frac{r}{r'}$ is always less than unity. This condition is fulfilled when we have $a(1+e) < a'(1-e')$. Writing g for $\varepsilon^{\gamma\sqrt{-1}}$, let
$$(r'^{-2}P)^{-\frac{2k+1}{2}} = \tfrac{1}{2}\sum_{j=-\infty}^{j=+\infty}.B^{(j)}_{\frac{2k+1}{2}}\,\varepsilon^{j(v-v'+\gamma)\sqrt{-1}}$$
$$= \tfrac{1}{2}\sum_{j=-\infty}^{j=+\infty}.B^{(j)}_{\frac{2k+1}{2}}\left(\frac{s-\omega}{1-\omega s}\right)^j\left(\frac{s'-\omega'}{1-\omega's'}\right)^{-j}g^j.$$

$B^{(j)}_{\frac{2k+1}{2}}$ is the same function of $\frac{r}{r'}$ that Laplace's $b^{(j)}_{\frac{2k+1}{2}}$ is of $\frac{a}{a'}=\alpha$. The approximate value of $\frac{r}{r'}$ being α, any function of $\frac{r}{r'}$ can be expanded in a series of ascending powers of $\frac{r}{r'}-\alpha$ by Taylor's Theorem. And as we have
$$\frac{r}{r'}-\alpha = \alpha\left\{\frac{\eta(1-\omega s)\left(1-\frac{\omega}{s}\right)}{\eta'(1-\omega's')\left(1-\frac{\omega'}{s'}\right)}-1\right\},$$
consequently
$$B^{(j)}_{\frac{2k+1}{2}} = \sum_{n=0}^{n=\infty}\frac{1}{n!}\alpha^n\frac{d^n b^{(j)}_{\frac{2k+1}{2}}}{d\alpha^n}\left\{\frac{\eta(1-\omega s)\left(1-\frac{\omega}{s}\right)}{\eta'(1-\omega's')\left(1-\frac{\omega'}{s'}\right)}-1\right\},$$
n being an integer, and $n!$ denoting the product of all integers up to n inclusive, it being understood that $0!=1$. Expanding the last factor of this expression by the binomial theorem, and employing the notation $[i,j]$ for the coefficient of x^j in the expansion of $(1+x)^i$, we have, p being an integer,

$$B^{(j)}_{\frac{2k+1}{2}} = \sum_{n=0}^{n=\infty}{}' \sum_{p=0}^{p=n} (-1)^{n-p} \frac{[n,p]}{n!} a^n \frac{d^n b^{(j)}_{\frac{2n+1}{2}}}{da^n} . \left\{ \frac{\eta(1-\omega s)\left(1-\frac{\omega}{s}\right)}{\eta'(1-\omega' s')\left(1-\frac{\omega'}{s'}\right)} \right\}^p .$$

13. In the next place the development of Q in terms of s and s' must be formed. We have

$$r \sin (v + \Pi) = \frac{1}{2\sqrt{-1}} \left[r\varepsilon^{(v+\Pi)\sqrt{-1}} - r\varepsilon^{-(v+\Pi)\sqrt{-1}} \right],$$

$$r' \sin (v' + \Pi') = \frac{1}{2\sqrt{-1}} \left[r'\varepsilon^{(v'+\Pi')\sqrt{-1}} - r'\varepsilon^{-(v'+\Pi')\sqrt{-1}} \right],$$

and putting $\quad \Pi + \Pi' = \theta, \qquad h = \varepsilon^{\theta\sqrt{-1}},$

we find

$$Q = - aa'\eta\eta' \sin^2\frac{I}{2} . \left[s\left(1-\frac{\omega}{s}\right)^2 - \frac{1}{s}\left(1-\omega s\right)^2 g^{-1}h^{-1} \right]$$
$$\times \left[s'\left(1-\frac{\omega'}{s'}\right)^2 h - \frac{1}{s'}\left(1-\omega' s'\right)^2 g \right].$$

Raising this expression to the k^{th} power, and multiplying by

$$r'^{-(2k+1)} = \left[a'\eta'(1-\omega' s')\left(1-\frac{\omega'}{s'}\right) \right]^{-(2k+1)},$$

we find that the part of $r'^{-(2k+1)}\, Q^k$ which has $h^{i'''}$ as a factor is

$$\frac{1}{a'} a^k \eta^k \eta'^{-(k+1)} \sin^{2k}\frac{I}{2} \sum_{n'=0}^{n'=k-i'''}{}' (-1)^{i'''} [k, n'] [k, k - i''' - n']$$
$$\times s^{2i'''-k+2n'} (1-\omega s)^{2k-2i'''-2n'} \left(1-\frac{\omega}{s}\right)^{2i'''+2n'}$$
$$\times s'^{k-2n'} (1-\omega' s')^{2n'-2k-1} \left(1-\frac{\omega'}{s'}\right)^{-2n'-1} g^{i'''-k+2n'} h^{i'''} .$$

14. We are now in the possession of all the developments necessary for exhibiting the function $\frac{1}{\Delta}$ in terms of s and s'. In order to obtain the part of this function which has $g^{i''}h^{i'''}$ for a factor, we must put, in the formulæ of § 12,

$$j = i'' - i''' + k - 2n',$$

and the chief operation here is the addition of the exponents of the quantities s, $1 - \omega s$, $1 - \frac{\omega}{s}$, and the similar functions of s' which are found in the three formulæ for $(r'^{-2}P)^{-\frac{2k+1}{2}}$, $B^{(j)}_{\frac{2k+1}{2}}$ and $r'^{-(2k+1)}\, Q^k$. For brevity we will write

$$[k] = \frac{1 . 3 \ldots (2k-1)}{2 . 4 \ldots . 2k}.$$

Then the part of $\frac{1}{\varDelta}$, which has $g^{i''}h^{i'''}$ for a factor, is

$$\frac{1}{2a'}\sum_{k=i'''}^{k=\infty}\sum_{n'=0}^{n'=k-i'''}\sum_{n=0}^{n=\infty}\sum_{p=0}^{p=n}(-1)^{k-i'''+n-p}\frac{[k][k,n'][k,k-i'''-n'][n,p]}{n!}$$

$$\times a^{k+n}\frac{d^n b_{\frac{2k+1}{2}}^{(i''-i'''+k-2n')}}{da^n}\sin^{2k}\frac{I}{2}$$

$$\times \eta^{k+p} s^{i''+i'''}(1-\omega s)^{k+p-i''-i'''}\left(1-\frac{\omega}{s}\right)^{k+p+i''+i'''}$$

$$\times \eta'^{-k-p-1} s'^{-i''+i'''}(1-\omega' s')^{-k-p-1+i''-i'''}\left(1-\frac{\omega'}{s'}\right)^{-k-p-1-i''+i'''} g^{i''}h^{i'''}.$$

We observe that in this expression the summation with respect to n' affects only the integral coefficients $[k, n']$, $[k, k-i'''-n']$ and the upper index of the quantity b, hence if a new function of a is assumed, which is a linear function of the b's, and such that

$$B_{\frac{2k+1}{2}}^{(i'',i''')}=\sum_{n'=0}^{n'=k-i'''}[k,n'][k,k-i'''-n']\,b_{\frac{2k+1}{2}}^{(i''-i'''+k-2n')}.$$

it will take the following simpler form

$$\frac{1}{2a'}\sum_{k=i'''}^{k=\infty}\sum_{n=0}^{n=\infty}\sum_{p=0}^{p=n}(-1)^{k-i'''+n-p}\frac{[k][n,p]}{n!}$$

$$\times a^{k+n}\frac{d^n B_{\frac{2k+1}{2}}^{(i'',i''')}}{da^n}\sin^{2k}\frac{I}{2}$$

$$\times \eta^{k+p} s^{i''+i'''}(1-\omega s)^{k+p-i''-i'''}\left(1-\frac{\omega}{s}\right)^{k+p+i''+i'''}$$

$$\times \eta'^{-k-p-1} s'^{-i''+i'''}(1-\omega' s')^{-k-p-1+i''-i'''}\left(1-\frac{\omega'}{s'}\right)^{-k-p-1-i''+i'''} g^{i''}h^{i'''}.$$

15. In order to get the coefficient of $z^i z'^{i'}$ in the expansion of $\frac{1}{\varDelta}$, according to the foregoing investigation we must multiply the preceding expression by $\quad \frac{r r'}{a a'}\,\varepsilon^{\frac{ie}{2}\left(s-\frac{1}{s}\right)+\frac{i'e'}{2}\left(s'-\frac{1}{s'}\right)}.$

Hence, if for brevity, we adopt the functional notation

$$S\begin{pmatrix} i-1\\ j\\ k\end{pmatrix}=\eta^i s^j(1-\omega s)^{i-j}\left(1-\frac{\omega}{s}\right)^{i+j}\varepsilon^{\frac{ke}{2}\left(s-\frac{1}{s}\right)},$$

$$S'\begin{pmatrix} i-1\\ j\\ k\end{pmatrix}=\eta'^i s'^j(1-\omega' s')^{i-j}\left(1-\frac{\omega'}{s'}\right)^{i+j}\varepsilon^{\frac{ke'}{2}\left(s'-\frac{1}{s'}\right)},$$

the coefficient of $z^i z^{i'} g^{i''} h^{i'''}$ in $\frac{1}{\varDelta}$ will be equal to the coefficient of $s^i s'^{i'}$ in

$$\frac{1}{2a'}\sum_{k=i'''}^{k=\infty}\sum_{n=0}^{n=\infty}\sum_{p=0}^{p=n}(-1)^{k-i'''+n-p}\,\frac{[k]\,[n,\,p]}{n\,!}$$

$$\times a^{k+n}\frac{d^n\mathfrak{B}^{(i'',\,i''')}_{\frac{2k+1}{2}}}{da^n}\sin^{2k}\frac{I}{2}\,.\ S\begin{pmatrix}k+p\\ i''+i'''\\ i\end{pmatrix}.\,S'\begin{pmatrix}-(k+p+1)\\ -(i''-i''')\\ i'\end{pmatrix}.$$

If then the coefficient of s^i in the expansion of S is denoted by $\mathfrak{E}$ followed by the same indices, and the coefficient of $s'^{i'}$ in the expansion of S' by $\mathfrak{E}'$ in like manner, $\mathfrak{E}$ will be a function of e only, and $\mathfrak{E}'$ a function of e' only; and, it being understood that each argument is taken but once, that is the negative of the argument is not considered, the coefficient of

$$\cos\,(\,i\zeta\,+\,i'\zeta'\,+\,i''\gamma\,+\,i'''\theta\,)$$

in the expansion of $\frac{1}{\varDelta}$ is expressed thus

$$\frac{1}{a'}\sum_{k=i'''}^{k=\infty}\sum_{n=0}^{n=\infty}\sum_{p=0}^{p=n}(-1)^{k-i'''+n-p}\,\frac{[k]\,[n,\,p]}{n\,!}$$

$$\times a^{k+n}\frac{d^n\mathfrak{B}^{(i'',\,i''')}_{\frac{2k+1}{2}}}{da^n}\sin^{2k}\frac{I}{2}\,.\ \mathfrak{E}\begin{pmatrix}k+p\\ i''+i'''\\ i\end{pmatrix}.\,\mathfrak{E}'\begin{pmatrix}-(k+p+1)\\ -(i''-i''')\\ i'\end{pmatrix}.$$

As in this formula k ought to be a positive integer, it will prevent embarrassment, if the arguments are so taken that i''' may not be negative. In the case where i, i', i'' and i''' are all zero, the expression must be divided by 2.

16. Thus we have arrived at an expression for the general coefficient involving only three signs of summation; and it may be remarked that all the coefficients are exhibited in precisely similar forms. Thus, to pass from one argument to an other, we have only to make the suitable changes in the two lower indices of the functions $\mathfrak{E}$ and $\mathfrak{E}'$ and in the upper indices of $\mathfrak{B}$, and commence the summation with reference to k with the new value of i''' instead of the old. Hence, from this expression, we can write out a scheme or blank form, which, when the indices proper to the argument are filled in, will be the coefficient of the cosine of it in the expansion of $\frac{1}{\varDelta}$. Such a blank form is written below; the indices i'' and i''' are omitted from $\mathfrak{B}$, and the two lower indices from $\mathfrak{E}$ and $\mathfrak{E}'$, and the upper indices of these quantities, for the sake of facility in writing, are placed to the right and at the foot. The factor $\frac{1}{a'}$, common to the whole expreson, is also omitted, so that the formula gives the coefficient in the expansion of $\frac{a'}{\varDelta}$. In making use of it, one must commence at the portion which has $\sin^{2i'''}\frac{1}{2}I$ for a factor, all the preceding parts being supposed to be suppressed. As to the double sign

before these factors, the upper or lower is taken according as i''' is even or odd. It is hoped that a sufficient number of terms have been written to render the law evident, so that they may be continued as far as desired.

$$
\begin{aligned}
& b_{\frac{1}{2}} E_0 E'_{-1} \\
& -\frac{1}{1}\alpha\frac{db_{\frac{1}{2}}}{d\alpha}\left[E_0E'_{-1} - E_1E'_{-2}\right] \\
& +\frac{1}{1.2}\alpha^2\frac{d^2b_{\frac{1}{2}}}{d\alpha^2}\left[E_0E'_{-1} - 2E_1E'_{-2} + E_2E'_{-3}\right] \\
& -\frac{1}{1.2.3}\alpha^3\frac{d^3b_{\frac{1}{2}}}{d\alpha^3}\left[E_0E'_{-1} - 3E_1E'_{-2} + 3E_2E'_{-3} - E_3E'_{-4}\right] \\
& + \ldots\ldots\ldots\ldots\ldots\ldots \\
\pm \frac{1}{2}\sin^2\frac{I}{2}\Bigg\{ & \alpha B_{\frac{3}{2}} E_1 E'_{-2} \\
& -\frac{1}{1}\alpha^2\frac{dB_{\frac{3}{2}}}{d\alpha}\left[E_1E'_{-2} - E_2E'_{-3}\right] \\
& +\frac{1}{1.2}\alpha^3\frac{d^2B_{\frac{3}{2}}}{d\alpha^2}\left[E_1E'_{-2} - 2E_2E'_{-3} + E_3E'_{-4}\right] \\
& -\frac{1}{1.2.3}\alpha^4\frac{d^3B_{\frac{3}{2}}}{d\alpha^3}\left[E_1E'_{-2} - 3E_2E'_{-3} + 3E_3E'_{-4} - E_4E'_{-5}\right] \\
& + \ldots\ldots\ldots\ldots\ldots\ldots \Bigg\} \\
\pm \frac{1.3}{2.4}\sin^4\frac{I}{2}\Bigg\{ & \alpha^2 B_{\frac{5}{2}} E_2 E'_{-3} \\
& -\frac{1}{1}\alpha^3\frac{dB_{\frac{5}{2}}}{d\alpha}\left[E_2E'_{-3} - E_3E'_{-4}\right] \\
& +\frac{1}{1.2}\alpha^4\frac{d^2B_{\frac{5}{2}}}{d\alpha^2}\left[E_2E'_{-3} - 2E_3E'_{-4} + E_4E'_{-5}\right] \\
& -\frac{1}{1.2.3}\alpha^5\frac{d^3B_{\frac{5}{2}}}{d\alpha^3}\left[E_2E'_{-3} - 3E_3E'_{-4} + 3E_4E'_{-5} - E_5E'_{-6}\right] \\
& + \ldots\ldots\ldots\ldots\ldots\ldots \Bigg\} \\
\pm \frac{1.3.5}{2.4.6}\sin^6\frac{I}{2}\Bigg\{ & \alpha^3 B_{\frac{7}{2}} E_3 E'_{-4} \\
& -\frac{1}{1}\alpha^4\frac{dB_{\frac{7}{2}}}{d\alpha}\left[E_3E'_{-4} - E_4E'_{-5}\right] \\
& +\frac{1}{1.2}\alpha^5\frac{d^2B_{\frac{7}{2}}}{d\alpha^2}\left[E_3E'_{-4} - 2E_4E'_{-5} + E_5E'_{-6}\right] \\
& -\frac{1}{1.2.3}\alpha^6\frac{d^3B_{\frac{7}{2}}}{d\alpha^3}\left[E_3E'_{-4} - 3E_4E'_{-5} + 3E_5E'_{-6} - E_6E'_{-7}\right] \\
& + \ldots\ldots\ldots\ldots\ldots\ldots \Bigg\} \\
\pm & \ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}
$$

For illustration, let it be desired to obtain the coefficient of

$$\cos(2\zeta - 5\zeta' + 2\gamma),$$

from which arises the larger part of the great inequality of Jupiter and Saturn, we have only to imagine that the lower indices $\begin{pmatrix} \cdot \\ 2 \\ 2 \end{pmatrix}$ are every where applied to E, and the indices $\begin{pmatrix} \cdot \\ -2 \\ -5 \end{pmatrix}$ to E', the indices $(2, 0)$ to B; and as we have $i''' = 0$, we suppress nothing and take the upper of the double signs.

17. The quantities B are very simply expressed in terms of the b's, the following are all that are needed when terms of the eighth order with respect to the inclination of the orbits are neglected.

$$
\begin{aligned}
B_{\frac{1}{2}}^{(i,0)} &= b_{\frac{1}{2}}^{(i)}, \\
B_{\frac{3}{2}}^{(i,0)} &= b_{\frac{3}{2}}^{(i+1)} + b_{\frac{3}{2}}^{(i-1)}, \\
B_{\frac{3}{2}}^{(i,1)} &= b_{\frac{3}{2}}^{(i)}, \\
B_{\frac{5}{2}}^{(i,0)} &= b_{\frac{5}{2}}^{(i+2)} + 4b_{\frac{5}{2}}^{(i)} + b_{\frac{5}{2}}^{(i-2)}, \\
B_{\frac{5}{2}}^{(i,1)} &= 2b_{\frac{5}{2}}^{(i+1)} + 2b_{\frac{5}{2}}^{(i-1)}, \\
B_{\frac{5}{2}}^{(i,2)} &= b_{\frac{5}{2}}^{(i)}, \\
B_{\frac{7}{2}}^{(i,0)} &= b_{\frac{7}{2}}^{(i+3)} + 9b_{\frac{7}{2}}^{(i+1)} + 9b_{\frac{7}{2}}^{(i-1)} + b_{\frac{7}{2}}^{(i-3)}, \\
B_{\frac{7}{2}}^{(i,1)} &= 3b_{\frac{7}{2}}^{(i+2)} + 9b_{\frac{7}{2}}^{(i)} + 3b_{\frac{7}{2}}^{(i-2)}, \\
B_{\frac{7}{2}}^{(i,2)} &= 3b_{\frac{7}{2}}^{(i+1)} + 3b_{\frac{7}{2}}^{(i-1)}, \\
B_{\frac{7}{2}}^{(i,3)} &= b_{\frac{7}{2}}^{(i)}.
\end{aligned}
$$

18. In computing the factors of the preceding formula which depend on E and E', the following abbreviation can be used. M_n denoting the factor which multiplies

$$\pm \frac{1}{n!} a^{k+n} \frac{d^n R_{\frac{2k+1}{2}}}{da^n},$$

and Δ being the symbol of finite differences with respect to n, it is plain that

$$\Delta^n M_n = (-1)^n E_{k+n} E'_{-(k+n+1)}.$$

Hence if the products $E_{k+n} E'_{-(k+n+1)}$ are computed for the various values of n, and are taken alternately with the positive and negative sign, and are written as if they were the successive differences of a function, we shall get the values of the factors M_n by filling out the scheme of differences. This

abbreviation is applicable equally whether we are making a numerical computation of the coefficient or a literal one. In the latter case the abbreviation can be applied separately to each term of the form $Ce^{i}e'^{i'}$ in the products $\mathcal{E}_k\mathcal{E}'_{-(k+1)}$.

19. We proceed now to discuss the functions $\mathcal{E}$. From their definition we have
$$\left(\frac{r}{a}\right)^i \varepsilon^{jv\sqrt{-1}} = \sum_{k=-\infty}^{k=+\infty} \mathcal{E}\begin{pmatrix} i \\ j \\ k \end{pmatrix} z^k,$$
whence
$$\left(\frac{r}{a}\right)^i \cos jv = \tfrac{1}{2}\sum_{k=-\infty}^{k=+\infty}\left[\mathcal{E}\begin{pmatrix} i \\ j \\ k \end{pmatrix} + \mathcal{E}\begin{pmatrix} i \\ -j \\ k \end{pmatrix}\right]\cos k\zeta,$$
$$\left(\frac{r}{a}\right)^i \sin jv = \tfrac{1}{2}\sum_{k=-\infty}^{k=+\infty}\left[\mathcal{E}\begin{pmatrix} i \\ j \\ k \end{pmatrix} - \mathcal{E}\begin{pmatrix} i \\ -j \\ k \end{pmatrix}\right]\sin k\zeta.$$
From which we gather that the functions $\mathcal{E}$ can be computed by definite integrals thus
$$\mathcal{E}\begin{pmatrix} i \\ j \\ k \end{pmatrix} = \frac{1}{\pi}\int_0^{\pi}\left(\frac{r}{a}\right)^i \cos(jv - k\zeta)d\zeta.$$
Let us now suppose that the coefficient of s^k, in the expansion of
$$\eta^i s^j(1-\omega s)^{i-j}\left(1-\frac{\omega}{s}\right)^{i+j}$$
in powers of s, is denoted by $E\begin{pmatrix} i \\ j \\ k \end{pmatrix}$, then evidently
$$\mathcal{E}\begin{pmatrix} i \\ j \\ k \end{pmatrix} = \sum_{l=-\infty}^{l=+\infty} . E\begin{pmatrix} i+1 \\ j \\ k-l \end{pmatrix} J^{(l)}_{\frac{ke}{2}}.$$
By writing in the expression $1 \div s$ for s and changing the sign of j, it remains unaltered; hence the relation
$$E\begin{pmatrix} i \\ -j \\ -k \end{pmatrix} = E\begin{pmatrix} i \\ j \\ k \end{pmatrix}.$$
By developing the factors of the expression
$$\eta^i s^j(1-\omega s)^{i-j}\left(1-\frac{\omega}{s}\right)^{i+j}$$
by the binomial theorem, we easily get
$$E\begin{pmatrix} i \\ j \\ k \end{pmatrix} = (-1)^{k-j}[i-j,\, k-j]\,\eta^i\,\omega^{k-j}$$
$$\text{t}\times\left[1 + \frac{(i+j)\,(i-k)}{1.(k-j+1)}\omega^2 + \frac{(i+j)(i+j-1)(i-k)(i-k-1)}{1.2.\,(k-j+1)\,(k-j+2)}\omega^4 + \ldots\right].$$
This equation, as written, is correct only when $k-j$ is not negative, but by h e relation given above we can reduce the case of $k-j$ negative to that where it is positive. The factor in the brackets is a case of the series
$$1 + \frac{\alpha\,.\,\beta}{1\,.\,\gamma}x + \frac{\alpha(\alpha+1)\,\beta(\beta+1)}{1\,.\,2\,.\,\gamma\,.\,(\gamma+1)}x^2 + \ldots$$

treated by Gauss in a memoir entitled "*Disquisitiones generales circa seriem infinitam &c.*" (See Gauss' Werke, Vol. III. p. 123, and especially the "Nachlass," p.207). According to Gauss' notation

$$E\begin{pmatrix} i \\ j \\ k \end{pmatrix} = (-1)^{k-j}[i-j,\, k-j]\,\eta^i\omega^{k-j}F(-i-j,\, k-i,\, k-j+1,\, \omega^2).$$

Whenever, of $i+j$ and $i-j$, one is not negative, this series terminates after a certain number of terms, thus affording a finite expression for the function. But when these integers are both negative, the series is infinite. However it can be easily transformed into another which like the former is finite. From Gauss' investigation of these series, (See the volume just quoted, p. 209, Equation [82]), we have

$$F(\alpha, \beta, \gamma, x) = (1-x)^{\gamma-\alpha-\beta}\, F(\gamma-\alpha, \gamma-\beta, \gamma, x).$$

Applying this to our expression, we get

$$E\begin{pmatrix} i \\ j \\ k \end{pmatrix} = (-1)^{k-j}[i-j,\, k-j]\eta^i\omega^{k-j}(1-\omega^2)^{2i+1}$$
$$\times F(i+k+1,\, i-j+1,\, k-j+1,\, \omega^2).$$

This expression is evidently finite when $i-j$ and $i+j$ are negative.

20. The developments of the functions $\mathcal{E}$ in powers of e as far as e^7 have been tabulated by Prof. Cayley in the Memoirs of the Royal Astronomical Society, Vol. XXVII. It would conduce to the ready employment of the preceding formulæ if we had the function $\mathcal{E}\begin{pmatrix} i \\ j \\ k \end{pmatrix}$ explicitly expanded in ascending powers of e, but the attempts I have made to write such a series lead to extremely complex forms of the coefficients. Hence I shall give here only the coefficient of the lowest power of e in this function, which suffices for obtaining all the terms of the lowest order in any coefficient of the expansion of $1 \div \Delta$. We have when $j-k$ is positive,

$$\mathcal{E}\begin{pmatrix} i-1 \\ j \\ k \end{pmatrix} = \Big[[i+j, j-k] + [i+j, j-k-1]\frac{k}{1} + [i+j, j-k-2]\frac{k^2}{1.2}$$
$$+ \ldots + [i+j, 0]\frac{k^{j-k}}{(j-k)!}\Big]\left(-\frac{e}{2}\right)^{j-k},$$

and when $k-j$ is positive,

$$\mathcal{E}\begin{pmatrix} i-1 \\ j \\ k \end{pmatrix} = \Big[[i-j, k-j] - [i-j, k-j-1]\frac{k}{1} + [i-j, k-j-2]\frac{k^2}{1.2}$$
$$- \ldots + [i-j, 0]\frac{(-k)^{k-j}}{(k-j)!}\Big]\left(-\frac{e}{2}\right)^{k-j}.$$

21. Thus in the example alluded to above of the coefficient of $\cos(2\zeta - 5\zeta' + 2\gamma)$, we find that the terms of the lowest order in E and E', (omitting here, as in the scheme, the two lower indices), are

$$E_0 = E_1 = E_2 = E_3 = 1,$$

$$E'_{-1} = -\left[[-2,3]-[-2,2]\frac{5}{1}+[-2,1]\frac{5^2}{1.2}-[-2,0]\frac{5^3}{1.2.3}\right]\left(\frac{e'}{2}\right)^3 = \frac{389}{48}e'^3.$$

$$E'_{-2} = -\left[[-3,3]-[-3,2]\frac{5}{1}+[-3,1]\frac{5^2}{1.2}-[-3,0]\frac{5^3}{1.2.3}\right]\left(\frac{e'}{2}\right)^3 = \frac{590}{48}e'^3.$$

$$E'_{-3} = -\left[[-4,3]-[-4,2]\frac{5}{1}+[-4,1]\frac{5^2}{1.2}-[-4,0]\frac{5^3}{1.2.3}\right]\left(\frac{e'}{2}\right)^3 = \frac{845}{48}e'^3.$$

$$E'_{-4} = -\left[[-5,3]-[-5,2]\frac{5}{1}+[-5,1]\frac{5^2}{1.2}-[-5,0]\frac{5^3}{1.2.3}\right]\left(\frac{e'}{2}\right)^3 = \frac{1160}{48}e'^3.$$

Bringing into use our method of abbreviation, we multiply each of the preceding numerical coefficients by 48 in order to avoid fractions, and then write them alternately with the positive and negative signs in a diagonal line, and from these, as successive orders of differences, derive the numbers standing in the vertical columns, thus,

$$\begin{array}{ccccccc}
+389 & & & & & & \\
 & -590 & & & & & \\
-201 & & +845 & & & & \\
 & +255 & & -1160 & & & \\
+54 & & -315 & & . & & \\
 & -60 & & +381 & & & \\
-6 & & +66 & & . & & \\
 & +6 & & -72 & & & \\
0 & & -6 & & . & & \\
 & 0 & & +6 & & & \\
 & & 0 & & . & & \\
 & & & 0 & & &
\end{array}$$

and dividing the numbers of the first column respectively by 1, — 1, 1 . 2, — 1.2.3, we get the following as the terms of the lowest order in the coefficient of $\cos(2\zeta - 5\zeta' + 2\gamma)$ in $a' \div \varDelta$,

$$\frac{1}{48}\left[389\, b_{\frac{1}{2}}^{(2)} + 201\, a\frac{db_{\frac{1}{2}}^{(2)}}{da} + 27a^2\frac{d^2b_{\frac{1}{2}}^{(2)}}{da^2} + a^3\frac{d^3b_{\frac{1}{2}}^{(2)}}{da^3}\right]e'^3,$$

which agrees with that found in the books. The following additional terms of the same coefficient can be written from the second, third, &c., columns, viz, those which are multiplied by e'^3 and the various powers of $\sin^2\frac{1}{2}I$,

$$-\frac{1}{2}\,\frac{1}{48}\left[590a\,B_{\frac{3}{2}}^{(2,0)} + 255a^2\frac{dB_{\frac{3}{2}}^{(2,0)}}{da} + 30a^3\frac{d^2B_{\frac{3}{2}}^{(2,0)}}{da^2} + a^4\frac{d^3B_{\frac{3}{2}}^{(2,0)}}{da^3}\right]e'^3\sin^2\frac{I}{2}$$

$$+\frac{1.3}{2.4}\,\frac{1}{48}\left[845a^2B_{\frac{5}{2}}^{(2,0)} + 315a^3\frac{dB_{\frac{5}{2}}^{(2,0)}}{da} + 33a^4\frac{d^2B_{\frac{5}{2}}^{(2,0)}}{da^2} + a^5\frac{d^3B_{\frac{5}{2}}^{(2,0)}}{da^3}\right]e'^3\sin^4\frac{I}{2}$$

— &c. . . . .

22. When we wish to obtain only the terms independent of ζ and ζ', that is those on which the secular perturbations depend, $i=0$ and $i'=0$, and the Besselian function J disappears from the expressions giving the values of E and E', and the coefficient of $\cos(i''\gamma + i'''\theta)$ in the expansion of $\frac{1}{\varDelta}$ can be written

$$\frac{1}{a'}\sum_{k=i'''}^{k=\infty}\sum_{n=0}^{n=\infty}\sum_{p=0}^{p=n}(-1)^{k-i'''+n-p}\frac{[k][n,p]}{n!}$$
$$\times a^{k+n}\frac{d^nB^{(i'',i''')}_{\frac{2k+1}{2}}}{da^n}\sin^{2k}\frac{I}{2}\cdot E\begin{pmatrix}k+p+1\\ i''+i'''\\ 0\end{pmatrix}\cdot E'\begin{pmatrix}-(k+p)\\ -(i''-i''')\\ 0\end{pmatrix}.$$

23. In leaving the subject of the development of $1\div\varDelta$, it may be well to note that two other forms can be given to the expression of the general coefficient, by employing, instead of the expression given above, either of the following

$$\frac{r}{r'}-a=a\frac{\frac{e'}{2}\left(s'+\frac{1}{s'}\right)-\frac{e}{2}\left(s+\frac{1}{s}\right)}{1-\frac{e'}{2}\left(s'+\frac{1}{s'}\right)}$$
$$=a\frac{\frac{e'}{2}\left(s'+\frac{1}{s'}\right)-\frac{e}{2}\left(s+\frac{1}{s}\right)}{\eta'(1-\omega's')\left(1-\frac{\omega'}{s'}\right)}.$$

But as they do not possess as much symmetry and brevity as the form given above, we will pass over them.

24. The second part of the Perturbative Function ,omitting the factor m', is

$$-\frac{r}{r'^2}\cos\psi=-\frac{r}{r'^2}\left[\cos^2\frac{I}{2}\cos(v-v'+\gamma)+\sin^2\frac{I}{2}\cos(v+v'+\theta)\right]$$
$$=-\frac{1}{2}\frac{r}{r'^2}\cos^2\frac{I}{2}\left[g\varepsilon^{(v-v')\sqrt{-1}}+g^{-1}\varepsilon^{-(v-v')\sqrt{-1}}\right]$$
$$-\frac{1}{2}\frac{r}{r'^2}\sin^2\frac{I}{2}\left[h\varepsilon^{(v+v')\sqrt{-1}}+h^{-1}\varepsilon^{-(v+v')\sqrt{-1}}\right].$$

According to the first theorem of § 3, the coefficient of z^0 in $\frac{r}{a}\varepsilon^{v\sqrt{-1}}$ is equal to that of s^0 in

$$\eta^2 s(1-\omega s)\left(1-\frac{\omega}{s}\right)^3;$$

or it is equal to $\quad -3\eta^2\omega(1+\omega^2)=-\tfrac{3}{2}e.$

And according to the second theorem, the coefficient of z^i in the same function is equal to that of s^i in

$$\frac{\eta}{i}\,\frac{d}{ds}\left[s\left(1-\frac{\omega}{s}\right)^2\right].\,s\varepsilon^{\frac{ie}{2}\left(s-\frac{1}{s}\right)}$$
$$=\frac{\eta}{i}\left(s-\frac{\omega^2}{s}\right)\varepsilon^{\frac{ie}{2}\left(s-\frac{1}{s}\right)}$$

Hence we have

$$\frac{r}{a}\,\varepsilon^{v\sqrt{-1}} = \sum_{i=-\infty}^{i=+\infty}\frac{\eta}{i}\left[J_{\frac{ie}{2}}^{(i-1)}-\omega^2 J_{\frac{ie}{2}}^{(i+1)}\right]z^i.$$

And by simply writing $1\div z$ for z,

$$\frac{r}{a}\,\varepsilon^{-v\sqrt{-1}} = \sum_{i=-\infty}^{i=+\infty}\frac{\eta}{i}\left[\omega^2 J_{\frac{ie}{2}}^{(i-1)}-J_{\frac{ie}{2}}^{(i+1)}\right]z^i.$$

The well known differential equations of elliptic motion

$$\frac{d^2x}{d\zeta^2}+\frac{a^3}{r^3}x=0,$$
$$\frac{d^2y}{d\zeta^2}+\frac{a^3}{r^3}y=0,$$

supposing the axis of x to be directed towards the perihelion, give us the equation

$$\frac{a^2}{r^2}\,\varepsilon^{v\sqrt{-1}}=-\frac{d^2\left(\frac{r}{a}\varepsilon^{v\sqrt{-1}}\right)}{d\zeta^2},$$

and consequently these two

$$\frac{a^2}{r^2}\varepsilon^{v\sqrt{-1}} = \sum_{i=-\infty}^{i=+\infty} i\eta\left[J_{\frac{ie}{2}}^{(i-1)}-\omega^2 J_{\frac{ie}{2}}^{(i+1)}\right]z^i,$$
$$\frac{a^2}{r^2}\varepsilon^{-v\sqrt{-1}} = \sum_{i=-\infty}^{i=+\infty} i\eta\left[\omega^2 J_{\frac{ie}{2}}^{(i-1)}-J_{\frac{ie}{2}}^{(i+1)}\right]z^i.$$

By substituting these values in the expression given above for $-\frac{r}{r'^2}\cos\psi'$, it is not difficult to see that, in it, the coefficient of

$$\cos\left(i\zeta+i'\zeta'+\gamma\right)$$

is $$-\frac{a}{a'^2}\cos^2\frac{I}{2}\,.\,\frac{\eta}{i}\left[J_{\frac{ie}{2}}^{(i-1)}-\omega^2 J_{\frac{ie}{2}}^{(i+1)}\right].\,i'\eta'\left[\omega'^2 J_{\frac{i'e'}{2}}^{(i'-1)}-J_{\frac{i'e'}{2}}^{(i'+1)}\right],$$

and the coefficient of

$$\cos\left(i\zeta+i'\zeta'+\theta\right)$$

is $$-\frac{a}{a'^2}\sin^2\frac{I}{2}\,.\,\frac{\eta}{i}\left[J_{\frac{ie}{2}}^{(i-1)}-\omega^2 J_{\frac{ie}{2}}^{(i+1)}\right].\,i'\eta'\left[J_{\frac{i'e'}{2}}^{(i'-1)}-\omega'^2 J_{\frac{i'e'}{2}}^{(i'+1)}\right].$$

In the special case of $i=0$ the middle factors of these expressions take the indeterminate form $0\div0$, but then, in accordance with what has been shown above, we should read $-\frac{3}{2}e$. Thus, by means of the Besselian functions, these coefficients take finite forms.

REMINISCENCES OF WILLIAM LENHART ESQ.

BY PROFESSOR DANIEL KIRKWOOD.

Dr. Hart's interesting "Historical Sketch of American Mathematical Periodicals" in the last No. of the ANALYST, makes honorable mention of my quondam friend, Mr. William Lenhart, the eminent mathematician. This reference has revived some of the most pleasant recollections of my youth. In 1838, when a tutor in the York County Academy, at York, Pennsylvania, I became acquainted with Mr. Lenhart, who was then, and had been for many years, an invalid, spending his summers in York, and his winters in Frederick, Maryland. He was a great enthusiast in his mathematical researches and having been without the sympathy of kindred minds, in York at least, he seemed gratified to find any one who was interested in his favorite studies. Mr. L., who was then over 50 years of age, related with much frankness the chief incidents of his eventful life, some of which may be found in the Princeton Review for July, 1841.

The mathematical talents of William Lenhart were first developed under the instruction of Mr. (afterward Dr.) Robert Adrain. About the year 1800 Mr. A., then a young man but little known, was chosen Principal of the York County Academy. William Lenhart, at the age of 13, became his pupil. The teacher at once noticed the splendid abilities of the student, and gave him more than ordinary attention and encouragement. William's father, however, had but little appreciation of mathematical learning, and withdrew him from school after an attendance of only 18 months. His subsequent attainments were entirely the result of his own unaided efforts

Dr. Hart in his list of writers for the *Mathematical Diary* names "Mary Bond, of Fredericktown, Md., the only female contributor." The solutions under that name, as well as one or two under that of *Diophantus*, were all by Mr. Lenhart. Of this fact I was informed by Mr. L. himself. He sometimes obtained quite different solutions of the same question, and in order to have both published sent one under an assumed signature.

Mr. Lenhart's library was very small. He showed me his books and manuscripts with great interest. Barlow's Theory of Numbers, The Ladies' Diary, from its commencement in 1704, The Mathematical Journals published in our own country, Euler's Algebra, and a few other mathematical works, are all now remembered. His manuscripts were the neatest I have ever seen, and some of them were handsomely bound in Morocco and gilt. He read to me a number of letters from Professor C. Gill, (Editor of the Mathematical Miscellany,) by whom the Diophantine speculations of Mr.

L. were published. In one of these letters it was affirmed that Mr. Lenhart had done more for the Diophantine Analysis than any other man that had ever lived. In another, Professor G., after expressing a desire to publish Lenhart's tables relating to squares and cubes, remarked that the printing would be attended with no inconsiderable expense. Mr. L., who was an inveterate smoker, at once, as he informed me, gave up his cigars, and appropriated the money thus saved to meet the expense of publication.

Of the distressing accident by which Mr. Lenhart, when a young man, was maimed for life, of the consequent blighting of his temporal prospects, and the intensity of his physical sufferings during a period of 28 years, it is unnecessary to speak, as the circumstances are detailed at some length in the article to which I have already referred. He died at Frederick, Md., on the 10th of July, 1840.

ON SUPPLYING OMISSIONS IN A CLOSED SURVEY.

BY CLEVELAND ABBE, WASHINGTON, D. C.

The problem of supplying any omissions in a closed survey which is treated by Prof. Philbrick in the ANALYST, p. 116, is one that I had occasion to explain more fully than is done in Gillespie's Treatise, while teaching the engineering students, in 1859, at Ann Arbor. A portion of my analytical solution of the problem was published, in 1861, in the Journal of the Franklin Institute, by Prof. DeVolson Wood with valuable additions by himself; and as it may be interesting to surveyors to look at the subject from this as well as the geometrical point of view, I will add the following to what Professors Wood and Philbrick have written.

Let the ordinary bearings (x) of the sides be converted into azimuthal angles by the following rules:

N. $x°$ *E.*	becomes azimuth	$x°$;	*S.* $x°$ *E.*	becomes	azimuth	$180° - x°$;
N. $x°$ *W.*	"	" $360° - x°$;	*S.* $x°$ *W.*	"	"	$180° + x°$;

the converse of which need not be repeated here.

Let the known sides be a_1, a_2, &c.; the known azimuths, b_1, b_2, &c.;
" unknown " " x_1, x_2, &c.; " unknown " y_1, y_2, &c.

The sides being always considered as positive numbers, the general equations of condition in a perfect survey are

$$\Sigma\ \text{Dep.} = a_1 \sin b_1 + a_2 \sin b_2 + a_3 \sin b_3\ \text{\&c.} = 0,$$

$$\Sigma\ \text{Lat.} = a_1 \cos b_1 + a_2 \cos b_2 + a_3 \cos b_3\ \text{\&c.} = 0.$$

From which any two quantities, if unknown, may be found by elimination.

The following are the six possible cases of omissions.

I. One distance is omitted. We find the required side from

$$x_1 = \frac{D}{\sin b_1} \qquad \text{or } x_1 = \frac{L}{\cos b_1},$$

where D and L indicate respectively the sums of the known Departures or Latitudes.

II. One bearing is omitted. The required azimuth is found from

$$\sin y_1 = \frac{D}{a_1} \qquad \text{or } \cos y_1 = \frac{L}{a_1}, \qquad \text{or } \tan y_1 = \frac{D}{L}.$$

In both these cases we have two equations for determining one unknown and it is best to compute them both. If the two resulting values are identical the numerical processes are correct.

III. The length and bearing of one side are omitted. The general equations give

$$\left.\begin{aligned} x_1 \sin y_1 &= D, \\ x_1 \cos y_1 &= L, \end{aligned}\right\} \text{whence } \tan y_1 = \frac{D}{L} \text{ and } x_1 = \frac{D}{\sin y_1} = \frac{L}{\cos y_1}.$$

IV. The length of one side and bearing of another are omitted. The general equations become

$$x_1 \sin b_1 + a_2 \sin y_2 = D,$$
$$x_1 \cos b_1 + a_2 \cos y_2 = L.$$

Whence $$\sin (y_2 - b_1) = \frac{1}{a_2}\Big(D \cos b_1 - L \sin b_1\Big).$$

This equation gives two values of $y_2 - b_1$ i. e. θ and $180° - \theta$, (whence result two values of y_2, i. e. $\theta + b_1$ and $180° + b_1 - \theta$, and therefore two values of x_1), so long as we have

$$D \cos b_1 - L \cos b_1 < a_2.$$

There will be but one solution when this inequality disappears, and there will be no possibility of closing the survey (indicating some error in the work of the surveyor or computer) if $D \cos b_1 - L \cos b_1 > a_2$.

V. The lengths of any two sides are omitted. We have in this case

$$x_1 \sin b_1 + x_2 \sin b_2 = D,$$
$$x_1 \cos b_1 + x_2 \cos b_2 = L;$$

whence we find x_1 and x_2 without ambiguity. One or the other of these lines may however result negative, and therefore the corresponding side of the survey will be imaginary.

VI. The bearings of any two sides are omitted. The general equations become

$$a_1 \sin y_1 + a_2 \sin y_2 = D,$$
$$a_1 \cos y_1 + a_2 \cos y_2 = L.$$

Introducing the auxilliary quantities m and θ such that

$$D = m \cos \theta \qquad \text{and } L = m \sin \theta$$

we have

$$\sin(\theta + y_2) = \frac{m^2 + (a_2^2 - a_1^2)}{2ma_2},$$

$$\sin(\theta + y_1) = \frac{m^2 - (a_2^2 - a_1^2)}{2ma_1};$$

whence are found two values for y_2 and y_1 respectively: these values of course become imaginary when the sine of $\theta + y_2$ or y_1 exceeds unity.

SOLUTION OF PROBLEM ON PAGE 40, NUMBER 2.

BY PROF. WILLIAM WOOLSEY JOHNSON, ANNAPOLIS, MD.

Two parabolas may be passed through four given points. The *directions* of their axes may be found by the following method given in Newton's Universal Arithmetic. Call the points A, B, C and D, join AB, BC, and CD; through A draw a parallel to CD, and through D a parallel to BC meeting the last parallel in M, and meeting AB in N. Construct a mean proportional to DM and DN, and lay it off in both directions from D on the line DM, the lines joining A with the points so found will be parallel to the axes of the two parabolas which may be passed through A, B, C, and D. [These directions may also be found by producing AB and CD to meet in O, laying off from O on OA a mean proportional to AO and OB, and on OD in both directions a mean proportional to OD and OC, the lines joining the points so found will have the required directions.]

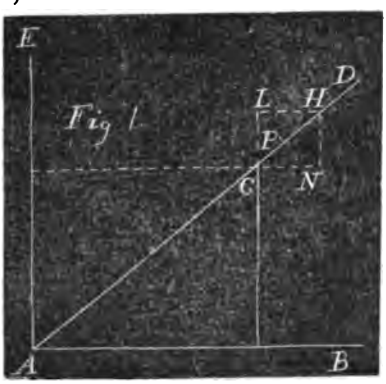
Having selected the direction of the axis, the solution may be completed as follows; produce BC to meet the diameter already drawn through A, produce AB to meet a diameter through D; join these points of intersection, and produce CD to meet the joining line, the line joining this last intersection with A will be a tangent to the required parabola. [This is an application of Pascal's Theorem, the hexagon being reduced to three chords, two diameters, or chords meeting the curve at infinity, and a tangent.] A tangent at one of the other points, say D, may now be drawn by bisecting the chord AD with a diameter, since the tangents at D and A will meet this diameter in the same point. Having now the tangents at A and D, lines making at A and D the same angles with these tangents which the latter make with the diameters, will meet in the focus. Hence the axis may be found; and, bisecting a subtangent, the vertex.

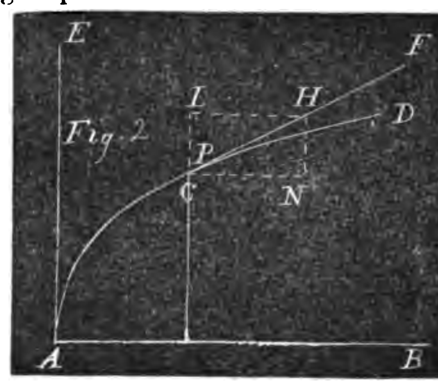
ON THE FIRST PRINCIPLES OF THE CALCULUS.

BY HIRAM COOK, C. E., NORWICH, CONN.

1. THE differential of a variable or function of a variable, is the *rate* at which such variable or function increases or decreases; that is, its *rate of variation*: and the object of the Differential Calculus is to determine the differential of a given function, in terms of the variable which enters such function, and the differential or rate of variation of that variable.

2. To illustrate the differential of a variable, that is, its rate of variation, we will suppose a particle P to be impelled from A toward D, along the line AD, and let the space described by P, at any instant, be represented by z; then the rate at which P is being impelled at the instant it coincides





with any point in the line AD, as C, will represent the rate that z is increasing at the same instant, or dz the differential of z.

It will be observed that as P is being impelled from A toward D, it recedes from the line AB; therefore, if y represents the distance of P from the line AB at right angles thereto, the rate at which P is receding from AB at any instant, as when it coincides with the point C, will represent the rate that y is increasing at the same instant, or dy the differential of y. Also, if x represents the distance of P from the line AE, a perpendicular to AB, the rate at which P is receding from AE at any instant, as when it coincides with the point C, will represent the rate at which x is incaeasing at that instant, or dx the differential of x.

Now if P is left to itself at any point as C, that is, if the force or forces which impell it from A along the line AD, are wthdrawn at C, it is obvious that P will thereafter proceed uniformly at the rate it is being impelled at C, and, in fig. 1, still in the direction of D, but in fig. 2, in the direction of F, along the line CF, tangent to the curve at C. Therefore, if

P, under the supposition that it is left to itself at C, describes the space CH during the first instant of time after leaving the point C, CH will represent the value of dz, CL or NH, of dy, and CN or LH, of dx; for it is evident that if P proceeds uniformly from C toward D, fig. 1, or from C toward F, fig. 2, at the rate it was being impelled at C, it will also recede uniformly from the line AB at the same rate it was receding at C, and likewise from the line AE. Consequently if P describes CH during the first instant of its motion from C, it will recede from AB the distance CL or NH, and from AE the distance CN or LH in the same time. Hence, since

$$CH^2 = CN^2 + NH^2,$$

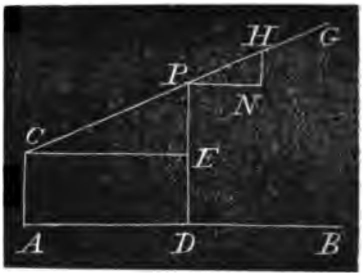
we have the following important equation; viz.,

$$dz^2 = dx^2 + dy^2.$$

3. Let the equation of the line CG be

$$y = ax + b \quad \ldots\ldots\ldots\ldots\ldots \quad (1)$$

in which x represents any abscissa, as AD, y the corresponding ordinate DP, and b the distance AC. Then, drawing CE parallel to AD, we have



$$DE = AC = b,$$
$$EP = DP - DE = y - b,$$

and $$CE = AD = x.$$

Therefore, if we represent dx by PN and dy by NH, Art. 1, we shall have

$$x : y - b :: dx : dy;$$

whence $$xdy = ydx - bdx,$$

or $$dy = \frac{ydx - bdx}{x}.$$

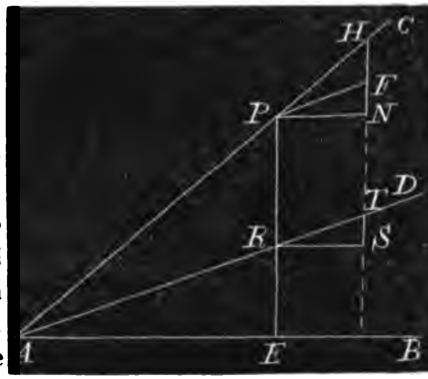
Substituting for y, in the second member of the last equation, its value from (1), we have

$$dy = \frac{axdx + bdx - bdx}{x},$$

or $$dy = adx,$$

the differential of the primitve function, $y = ax + b$.

It will be seen that if $b = 0$, the differential will be the same. Hence:—*A constant quantity has no differential; that is, a constant term, connected with a variable by the sign plus or minus, will disappear in differentiating. Also, the differential of a variable multiplied by a constant quantity, is equal to the constant into the differential of the variable.*



4. If any ordinate of the line AC, as EP, is represented by u, and ER, that portion of EP which is the ordinate of the line AD, by y; then, as has been shown, Art. 1, du may be represented by NH, and dy by ST.

Now if PF is drawn parallel to RT, NF will be equal to ST, and consequently NH, the representation of the rate of increase of EP, will exceed ST, the representation of the rate of increase of ER, by FH: hence FH will represent the rate of increase of RP; that is, if RP is represented by z, dz will be represented by FH. Therefore, since

$$EP = u = ER + RP = y + z,$$

and $$NH = du = NF + FH = ST + FH = dy + dz,$$

we have $$du = dy + dz \quad \ldots\ldots\ldots\ldots\ldots \quad (2)$$

for the differential of $$u = y + z.$$

If we transpose (2), we have $dy = du - dz$, which is the differential of

$$y = u - z,$$

as will be seen in the figure.

Hence:—*The differential of the sum or difference of two variables, is equal to the corresponding sum or difference of their differentials.*

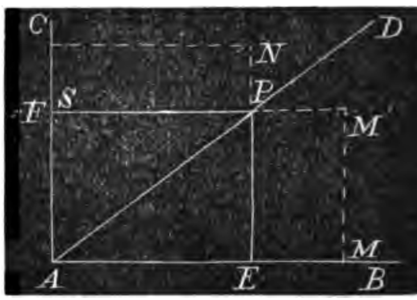
5. Let AC (see Fig. on next page) be at right angles to AB, and suppose a point P to advance from A toward D, along the line AD, carrying with it the lines PR and PS, PR perpendicular to AB and PS perpendicular to AC; and suppose these lines so to increase in length, that R, the extremity of one, shall always be in the line AB, and S, the extremity of the other, shall always be in the line AC. Then, at any instant, as when R and S simultaneously arrive at the points E and F in the lines AB and AC, the line PR will have described the area APR, and the line PS the area APS.

Now it is evident that a right line, advancing in a direction at right angles to itself, will, at any instant, be describing an area equal to the length of the line at that instant, multiplied by the rate at which it is then advancing. Thus, if a line at the instant it is ten inches long, is advancing at the rate of two inches per second, it will then be describing an area at the rate of two times ten, or twenty square inches per second.

Hence, if RM or PM' represents the rate at which PR is advancing at the instant R arrives at E, the rectangle $RPM'M$ will represent the rate at

which PR is then describing area, or the rate at which the area APR is then increasing. So also, if SN or PN' represents the rate at which PS is advancing at the instant S arrives at F, the rectangle $SNN'P$ will represent the rate at which the area APS is increasing at the same instant, viz., at the instant S arrives at F.

But if x represents the distance of R from A at any instant, and y the distance of S from A at the same instant; then, Art. 1, RM will be represented by dx and SP by dy. Therefere, since RP is eqnal to y and SP to x, ydx represents the rectangle RP $M'M$, and xdy the rectangle $SNN'P$; and consequently $xdy + ydx$ represents the rate at which the area of both APS and APR, or the area of the rectangle $ASPR$, is increasing at the instant R and S simultaneously arrive at the points E and F.



If we reprsent the area of the rectangle $ASPR$ by u, then, since the sides of the rectangle are represented by x and y, we have

$$u = xy. \qquad (3)$$

But, since u represents the area, du the differential of u, must represent the rate of increase of the area, which has been shown to be $xdy + ydx$; therefore

$$du = xdy + ydx. \qquad (4)$$

Hence:—*The differential of the product of two variables is equal to the first into the differential of the second, plus the second into the differential of the first.*

6. If in (3) we make y equal to x, u will be equal to x^2; also dy will be equrl to dx, for if y and x are always equal their rates of increase must necessarily be equal. Therefore, by substituting x for y and dx for dy in (4), we find the differential of

$$u = x^2$$

to be
$$du = xdx + xdx = 2xdx;$$
that is,
$$du = 2x^{2-1}dx.$$

If we make $y = x^2$, then will $dy = 2xdx$ and $u = x^3$. Therefore, by substituting x^2 for y and $2xdx$ for dy in (3), we find the differential of

$$u = x^3$$

to be
$$du = 2x^2dx + x^2dx = 3x^2dx;$$
or
$$du = 3x^{3-1}dx.$$

Again, if we make $y = x^3$, then will $dy = 3x^2dx$ and $u = x^4$. Therefore, by substituting in (3) x^3 for y, and $3x^2dx$ for dy, we find the differential of

$$u = x^4$$

to be $du = 3x^3dx + x^3dx = 4x^3dx;$

or $du = 4x^{4-1}dx.$

Proceeding in this manner, we shall find the differentials of

$u = x^5$ to be $du = 5x^{5-1}dx,$
$u = x^6$ " " $du = 6x^{6-1}dx,$
$u = x^7$ " " $du = 7x^{7-1}dx;$

from which it is evident that the differential of

$u = x^n$, in which n is any positive whole number, is $du = nx^{n-1}dx.$

Therefore the differential of $u = ax^n$ (by Art. 3) is

$$du = anx^{n-1}dx.$$

Hence:— *The differential of any positive integral power of a variable multiplied by a constant quantity, is equal to the constant into the product of the exponent denoting the power and the variable with its exponent diminished by unity, multiplied by the differential of the variable.*

Proceeding in like manner, the differentials of all algebraic functions may readily be determined.

7. Let us now consider the exponential function

$$u = a^x.$$

Assuming $a = 1 + c$ we have $u = (1 + c)^x$, or, developing $(1 + c)^x$ by the binomial theorem,

$$u = 1 + xc + \frac{x(x-1)}{2}c^2 + \frac{x(x-1)(x-2)}{2.3}c^3 + \&c. \quad . . . (5)$$

Differentiating this equation we obtain

$$du = \left(c + \frac{2x-1}{2}c^2 + \frac{3x^2 - 6x + 2}{2.3}c^3 + \&c.\right)dx:$$

Dividing each member of this equation by the corresponding member of (5), we have

$$\frac{du}{u} = \left(c - \frac{c^2}{2} + \frac{c^3}{3} - \frac{c^4}{4} + \&c.\right)dx.$$

Substituting for u its value a^x, and for c, its value $a - 1$, and multiplying by a^x, we find

$$du = a^x\left(\frac{a-1}{1} - \frac{(a-1)^2}{2} + \frac{(a-1)^3}{3} - \frac{(a-1)^4}{4} + \&c.\right)dx;$$

but the series within the parenthesis, in this equation, is the Napierian logarithm of a, (see Brande, Art. Logarithm), therefore we have

$$du = a^xdx \log. a.$$

Hence:—*The differential of an exponential function, is equal to the Napierian logarithm of the constant of which the variable is the exponent, multiplied by the function itself into the differential of the exponent.*

8. Resuming the given function $u = a^x$ of the preceding article and taking the logarithm of both members, we have

$$\log u = x \log a, \qquad \text{whence}$$

$$x = \frac{\log u}{\log a} \ldots\ldots\ldots\ldots\ldots\ldots, \quad (6)$$

Now, Art. 7, the differential of $u = a^x$ is $du = a^x dx \log a$, whence

$$dx = \frac{du}{a^x \log a}, \text{ or, substituting } u \text{ for } a^x, \; dx = \frac{du}{u \log a} \ldots\ldots \quad (7)$$

Therefore, (7) is the differential of the logarithmic function (6).

If a is the base of a system of logarithms, then x is the logarithm of u in that system and $1 \div \log a$ is the modulus of the system. Therefore, if we represent $1 \div \log a$, in (6), by M, we have

$$dx = M\frac{du}{u} \ldots\ldots\ldots\ldots\ldots\ldots \quad (8)$$

Hence:—*The differential of the logarithm of a variable is equal to the modulus of the system into the differential of the variable, divided by the variable.*

9. The modulus of the Napierian system is unity; therefore, if the logarithms are taken in that system (8) becomes

$$dx = \frac{du}{u};$$

that is, the differential of the Napierian logarithm of a variable, is the differential of the variable divide by the variable.

By the preceding method, all exponential and logarithmic functions may be differentiated.

10. We will now consider the trigonometrical function $u = \sin x$, in which x is the arc and u its sine.

Let $AB = x$, $BD = u$, and $CB = R$; then, Art. 2, we have BF represented by dx and EF by du. Therefore, since the angle EBF is equal to the angle CBD and $CD = \cos x$, we have

$$R : \cos x :: dx : du,$$

or

$$du = \frac{\cos x \, dx}{R}.$$

Hence:—*The differential of the sine of an arc is equal to the cosine of the arc, into the differential of the arc, divided by the radius.*

In a similar manner we may find the following functions to have the respective differentials, viz.;

$$u = \cos x, \quad du = -\frac{\sin x \, dx}{R}; \qquad u = \text{ver.sin}\, x, \quad du = \frac{\sin x \, dx}{R};$$

$$u = \tan x, \quad du = \frac{R^2 \, dx}{\cos^2 x}; \qquad u = \cot x, \quad du = -\frac{R^2 \, dx}{\sin^2 x};$$

$$u = \sec x, \quad du = \frac{\tan x \, dx}{\cos x}; \qquad u = \text{cosec}\, x, \quad du = -\frac{\cot x \, dx}{\sin x}.$$

11. To determine the differential of the logarithmic sine of an arc, or of the function $$u = \log \sin x.$$
Differentiating, Arts. 9 and 10, we find
$$du = \frac{\cos x\, dx}{R \sin x}, \text{ or, since } \frac{\cos x}{R \sin x} = \frac{1}{\tan x},\ du = \frac{dx}{\tan x}.$$

In a similar manner we shall find the differentials of $u = \log \cos x$, and $u = \log \tan x$, to be, respectively,
$$du = -\frac{dx}{\cot x}, \qquad \text{and } du = \frac{\tan x\, dx}{\sin^2 x}.$$

12. Finally we will take the function $z = \sin^{-1} x$, ... (8) in which z is the arc of which x is the sine. Hence, the equivalent relation will be $x = \sin z$; the differential of which, Art. 10, is
$$dx = \frac{\cos z\, dz}{R}; \text{ whence we have } dz = \frac{R\, dx}{\cos z}.$$
But since $\sin z = x$, and consequently $\cos z = \sqrt{(R^2 - x^2)}$, we have
$$dz = \frac{R\, dx}{\sqrt{(R^2 - x^2)}},$$
which is the differential of (8); i. e., the differential of the arc in functions of its sine. Similarly we shall find the differentials of $z = \cos^{-1} x$, $z = \tan^{-1} x$ &c., to be, respectively,
$$dz = -\frac{R\, dx}{\sqrt{(R^2 - x^2)}}, \qquad dz = \frac{R^2 dx}{R^2 + x^2}, \text{ \&c.}$$

Upon these principles the whole structure of the Differential Calculus may be erected so as to be readily comprehended by the ordinary student.

SOLUTIONS OF PROBLEMS IN NUMBER FIVE.

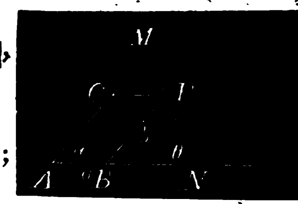
Solutions of problems in No. five have been received as follows:

From Marcus Baker, 83, 84 & 85; E. S. Farrow, 84, 85, 86, 87 & 88; J. M. Greenwood and W. H. Baker, 83, 84, 86, 87 & 88; Henry Gunder, 84 & 86; Prof. W. W. Johnson, 86 & 91; Prof. H. T. J. Ludwick, 84, & 86; F. P. Matz, 86; Dr. A. B. Nelson, 84, 86, 87 & 88; O. D. Oathout, 84; Walter Siverly, 83, 84, 86, 87, 88, 89, 90 & 91; E. B. Seitz, 83 84 & 88; Prof. J. Scheffer, 83, 84, 85, 87 & 88; Prof. D. Trowbridge, 86; R. J. Adcock, 89.

83. "A point is given within two lines which form a given angle with one another. Required the shortest line which can be drawn through this point, terminated by the given lines."

SOLUTION BY M. BAKER, U. S. COAST SURVEY.

From the Fig. $PN = b \sin a \operatorname{cosec} \theta$ and $PM = a \sin a \operatorname{cosec}(a + \theta)$. $PM + PN = MN = \sin a\,[b \operatorname{cosec}\theta + a \operatorname{cosec}(a + \theta)]$, a minimum.



Differentiating we have $b \cot\theta \operatorname{cosec}\theta + a \cot(a + \theta) \operatorname{cosec}(a + \theta) = 0$; whence

$$a \cot(a + \theta) = -b\frac{\sin(a + \theta)}{\tan\theta \sin\theta},$$

$$a\frac{1 - \tan a \tan\theta}{\tan a + \tan\theta} = -b\frac{\sin a \cos\theta + \cos a \sin\theta}{\tan\theta \sin\theta} = -b\frac{\sin a + \cos a \tan\theta}{\tan^2\theta},$$

$a \tan^2\theta - a \tan a \tan^3\theta = -b \sin a \tan a - 2b \sin a \tan\theta - b \cos a \tan^2\theta$, and finally

$$\tan^3\theta - \frac{a + b\cos a}{a \tan a}\tan^2\theta - \frac{2b \sin a}{a \tan a}\tan\theta - \frac{b}{a}\sin a = 0,$$

from which θ may be determined. When $a = 90^\circ$ $\tan\theta = \sqrt[3]{(b \div a)}$.

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84. "The centres of two spheres whose radii are 12 ft. and 5 ft., respectively, are at opposite extremities of the diameter of a circle of 13 ft. radius. Find a point in the circumference of this circle from which the greatest portion of spherical surface is visible.

SOLUTION BY PROF. H. T. J. LUDWICK, SALISBURY, NORTH CAROLINA.

Let $x =$ the distanc of the point from the centre of the larger sphere. Put $a = 12$, $b = 5$, and $d = 13$.

Of the two segments seen, the height of the lager is found $= a - a^2 \div x$, and the height of the smaller $= b - b^2 \div \sqrt{(4d^2 - x^2)}$.

$\therefore$ Surface of segment of larger sphere $= 2\pi a\left(a - \dfrac{a^2}{x}\right)$,

and " " " " smaller " $= 2\pi b\left(b - \dfrac{b^2}{\sqrt{(4d^2 - x^2)}}\right)$.

Hence their sum $S = 2\pi a\left(a - \dfrac{a^2}{x}\right) + 2\pi b\left(b - \dfrac{b^2}{\sqrt{(4d^2 - x^2)}}\right)$.

$$\therefore \quad \frac{dS}{dx} = \frac{a^3}{x^2} + \frac{b^3 x}{\sqrt{[(4d^2 - x^2)^3]}} = 0.$$

$$\therefore \quad x = \frac{2ad}{\sqrt{(a^2 + b^2)}} = \frac{2 \,.\, 12 \,.\, 13}{\sqrt{(12^2 + 5^2)}} = 24;$$

and $26^2 - 24^2 = 10^2$. Thereforer the point is 10 feet from the centre of the smaller sphere.

85. "In a quadrilateral there are given, the length and position of the lower base, the lengths of the two sides, the length of the upper base and the position of a point through which it passes: required to construct the quadrilateral."

[No *construction* of this prob. has been received.—Prof. J. Scheffer writes: "If we describe a circle about one of the extrmities of the lower base of the quadrilateral as a centre, with a radius equal to one of the sides, and another circle about the other extremity of the lower base as a centre, with a radius equal the other side, we reduce the problem to the following:

If two circles are given as to magnitude and position, to lay a line of given length between the two circumferences, which passes through a given point.

The line of given length is of course the upper base. As the line may be laid between the outer circumferences, or one outer and one inner circumference, or two inner circumferences, the problem admits in general of four solutions.

This problem belongs to those which cannot be solved by means of the straight line and circle only, and, therefore, exceeds the power of Plane Geometry."

Marcus Baker says that in the solution of 85 the equation is involved to the 8th degree.]

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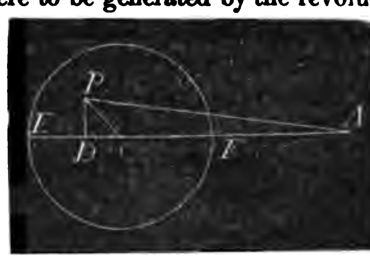
86. "Prove that the attraction of a sphere of uniform density upon an external point is the same as if all the matter of the sphere were concentrated at its centre."

SOLUTION BY A. B. NELSON M. D., DANVILLE, KY.

Let A be the point, and suppose the sphere to be generated by the revolution of a semicircle about its diameter EF. Take the origin of coordinates at the centre C, and let CD and PD be the coordinates of any point P of the semicircle. Then the elemental ring of matter generated by the particle P, $= 2\pi y dy dx$.



Denote CF by 1, and AC by a. The limits of x are $+1$ and -1; and of y, 0 and $\sqrt{(1-x^2)}$. The attraction of an elemental ring upon A, measured along CA is

$$\frac{2\pi y dy dx}{AP^2}\times\frac{AD}{AP}=\frac{2\pi y dy dx(a+x)}{[(a+x)^2+y^2]^{\frac{3}{2}}}.$$

Hence the sum of the attractions of all the particles of the sphere is

$$2\pi\int_{-1}^{+1}(a+x)dx\int_{0}^{\sqrt{(1-x^2)}}\frac{ydy}{[(a+x)^2+y^2]^{\frac{3}{2}}}$$

$$=2\pi\int_{-1}^{+1}(a+x)dx\left(\frac{1}{a+x}-\frac{1}{(a^2+2ax+1)^{\frac{1}{2}}}\right)=\frac{4\pi}{3a^2},$$

which is equal to the attraction of the sphere at the distance a, if all its particles were concentrated at the centre.

[For solution of this problem Mr. Siverly refers to the following works: Simpson's Fluxions p. 450, Emerson's Fluxions p. 370, Vince's Fluxions p. 116, Dealtry's Fluxions p. 146, Poisson's Mechanics Art. 100, Mecanique Celeste Book II, Chap. 2, and Earnshaw's Dynamics p. 333. To which may be added, Newton's Principia Book I, Prop. LXXI.]

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87. "There are n tickets in a bag numbered 1, 2, 3, . . . n. A man draws three tickets together at random and is to receive a number of shillings equal to the product of the numbers he draws. Find the value of his expectation."

SOLUTION BY J. M. GREENWOOD AND W. H. BAKER, KANSAS CITY, MO.

Let s be the sum of the products taken three in a set, p the number of products, and v the value of the chance.

Then $$v=\frac{s}{p} \text{ and } p=\frac{n(n-1)(n-2)}{6}. \quad \ldots\ldots\ldots\ldots (1)$$

To find s, (see Todhunter's Algebra, Art. 227), we have

$$(1+2+3+\ldots n)^3=1^3+2^3+3^3+\ldots n^3+3.1^2(2+3+\ldots n)$$
$$+3.2^2(1+3+4+\ldots n)+\ldots\ldots 3.n^2(1+2+3+\ldots n-1)+6s\ldots(2)$$

But $$1+2+3+4+\ldots n=\frac{n(n+1)}{2}, \quad \ldots\ldots\ldots (3)$$

$$1^3+2^3+3^3+4^3+\ldots n^3=\frac{n^2(n+1)^2}{4}, \quad \ldots\ldots\ldots (4)$$

$$3.1^2(2+3+\ldots n)+3.2^2(1+3+4+\ldots n)+\ldots 3.n^2(1+2+3+\ldots n-1)$$
$$=3\,1^2\left(\frac{n(n+1)}{2}-1\right)+3.2^2\left(\frac{n(n+1)}{2}-2\right)+\ldots 3n^2\left(\frac{n(n+1)}{2}-n\right)$$
$$=3.\frac{n(n+1)}{2}.(1^2+2^2+3^2+\ldots n^2)-3(1^3+2^3+3^3+\ldots n^3)$$
$$=\frac{3n^2(n+1)}{2}.\frac{n(n+1)(2n+1)}{6}-\frac{3n^2(n+1)^2}{4}\ldots\ldots\ldots(5)$$

Substituting in (2)

$$\left(\frac{n(n+1)}{2}\right)^3=\frac{n^2(n+1)^2(2n+1)}{4}-\frac{n^2(n+1)^2}{2}+6s.\ \ldots\ldots(6)$$

Whence $s=\dfrac{n^2(n+1)^2(n-1)(n-2)}{48}$; $\therefore\ v=\dfrac{s}{p}=\dfrac{n(n+1)^2}{8}$.

88. "An ellipse revolves about its latus rectum; show that the volumes of the solids generated by the larger and smaller segments are respectively equal to" &c.

SOLUTION BY E. B. SEITZ, GREENVILLE, OHIO.

The equation to the ellipse referred to the latus rectum and major axis, is

$$y = ae \pm \sqrt{\left(a^2 - \frac{x^2}{1-e^2}\right)}.$$

Hence the volume of the solid generated by the larger segment is

$$2\pi \int_0^{a(1-e^2)} \left[ae + \sqrt{\left(a^2 - \frac{x^2}{1-e^2}\right)}\right]^2 dx + 2\pi \int_{a(1-e^2)}^{a(1-e^2)^{1/2}} \left\{ \left[ae + \sqrt{\left(a^2 - \frac{x^2}{1-e^2}\right)}\right]^2 - \left[ae - \sqrt{\left(a^2 - \frac{x^2}{1-e^2}\right)}\right]^2 \right\} dx$$

$$= \frac{4\pi a^3}{3}\left(1-e^2\right)\left[\frac{2+e^2}{2} + \frac{3e}{(1-e^2)^{1/2}} \tan^{-1}\left(\frac{1+e}{1-e}\right)^{\frac{1}{2}}\right];$$

and the volume of the solid generated by the smaller segment is

$$2\pi \int_0^{a(1-e^2)} \left[ae - \sqrt{\left(a^2 - \frac{x^2}{1-e^2}\right)}\right]^2 dx$$

$$= \frac{4\pi a^3}{3}\left(1-e^2\right)\left[\frac{2+e^2}{2} - \frac{3e}{(1-e^2)^{1/2}} \tan^{-1}\left(\frac{1-e}{1+e}\right)^{\frac{1}{2}}\right].$$

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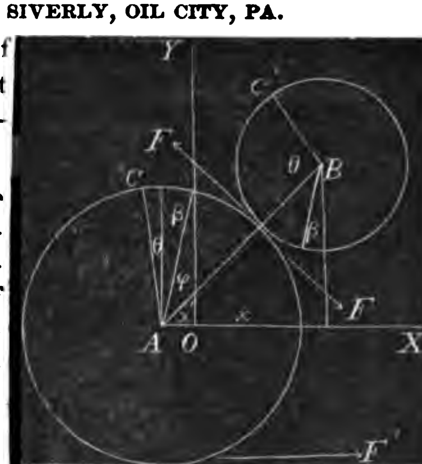
89. "A sphere, radius r, *rolls* down the surface of another sphere of the same material, radius R, placed on a horizontal plane. The surfaces of both spheres and plane are rough enough to secure perfect rolling. Determine the motion of the spheres, the point of separation and the equation of the curve described by the center of the upper sphere."

SOLUTION BY WALTER SIVERLY, OIL CITY, PA.



It is evident from the principle of the motion of the center of gravity that the spheres will roll in opposite directions.

Let A and B be the centers of the spheres at any time t after the beginning of the motion, O being the initial position of A; D the point of contact of the spheres; C, C' the points that were in contact at the beginning of the motion; m, m', the masses of A and B respectively; O the origin of horizontal and vertical

coordinates; $(-x, 0)$, (x, y) the coordinates of A and B; F the friction between the spheres, F' the friction between the lower sphere and the plane; θ, θ' the angles through which the spheres have respectively revolved, φ the inclination of AB to the vertical, β being the initial value of φ.

Since F' is the only force acting on the system horizontally,

$$m'\frac{d^2x}{dt^2} - m\frac{d^2x'}{dt^2} = F'. \quad \ldots\ldots\ldots\ldots (1).$$

The rotation of the spheres gives

$$\frac{2}{5} m R \frac{d^2\theta}{dt^2} = F' - F, \quad \ldots\ldots\ldots\ldots (2)$$

$$\frac{2}{5} m' r \frac{d^2\theta}{dt^2} = F. \quad \ldots\ldots\ldots\ldots\ldots (3)$$

Since there is perfect rolling,

$$R(\varphi + \theta - \beta) = r(\theta' - \varphi + \beta), \quad \ldots\ldots (4)$$

$$x' = R\theta. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

Also,

$$x + x' = (R + r)\sin\varphi, \quad \ldots\ldots\ldots\ldots (6)$$

$$y = (R + r)\cos\varphi. \quad \ldots\ldots\ldots\ldots (7)$$

By the principle of *vis viva*,

$$m\left(\frac{2}{5}R^2\frac{d\theta^2}{dt^2} + \frac{dx'^2}{dt^2}\right) + m'\left(\frac{2}{5}r^2\frac{d\theta'^2}{dt^2} + \frac{dx^2}{dt^2} + \frac{dy^2}{dt^2}\right) = 2m'g(c - y)$$

$$= 2m'g(R + r)(\cos\beta - \cos\varphi). \quad \ldots\ldots\ldots (8)$$

Eliminating F, F' from (1), (2), (3) and integrating once,

$$\frac{2}{5}m R \frac{d\theta}{dt} + \frac{2}{5} m' r \frac{d\theta'}{dt} = m'\frac{dx}{dt} - m\frac{dx'}{dt}. \quad \ldots\ldots\ldots (9)$$

Eliminating $\frac{dx}{dt}$, $\frac{dx'}{dt}$, $\frac{dy}{dt}$, $\frac{d\theta}{dt}$, $\frac{d\theta'}{dt}$ from (8), (9), by (4), (5), (6), (7),

$$\left(49m + 45m' - 25m'\cos^2\varphi + 20m'\cos\varphi\right)\frac{d\varphi^2}{dt^2} = \frac{70g}{R+r}\left(m+m'\right)\left(\cos\beta - \cos\varphi\right) \quad \ldots (10)$$

At the point of separation each sphere moves uniformly horizontally, hence

$$\frac{d^2x}{dt^2} + \frac{d^2x'}{dt^2} = 0.$$

Differentiating (6) once,

$$\frac{dx}{dt} + \frac{dx'}{dt} = (R + r)\cos\varphi . \frac{d\varphi}{dt}.$$

Substituting $\frac{d\varphi}{dt}$ from (10), differentiating and putting $\frac{d^2x}{dt^2} + \frac{d^2x'}{dt^2} = 0$, we obtain $25m'\cos^3\varphi - 40m'\cos^2\varphi - [3(49m+45m') - 20m'\cos\beta]\cos\varphi$

$$= -2\cos\beta\,(49m + 45m'),$$

which determines the point of separation.

Differentiating (6), (7) once and substituting for x' its value from (5),

$$dx + Rd\theta = (R + r)\cos\varphi d\varphi, \qquad (11)$$
$$dy = \quad -(R + r)\sin\varphi d\varphi. \qquad (12)$$

Eliminating θ', x' from (9) by (4), (5), and cancelling dt,

$$(7m + 2m')R d\theta + 2m'(R + r)\,d\varphi = 5m'dx. \qquad (13)$$

Eliminating $d\theta$ by (11), then $d\varphi$ by (12) and substituting for $\sin\varphi$, $\cos\varphi$, their values $\frac{\sqrt{[(R + r)^2 - y^2]}}{R + r}$, $\frac{y}{R + r}$,

$$7(m + m')\,dx = (-7m + 2m')\frac{ydy}{\sqrt{[(R + r)^2 - y^2]}} - \frac{2m'(R + r)\,dy}{\sqrt{[(R + r)^2 - y^2]}}.$$

Integrating and observing that initially $x = (R+r)\sin\beta$, $y = (R+r)\cos\beta$,

$$7(m+m')x = (7m+2m')\sqrt{[(R+r)^2 - y^2]} + 5m'(R+r)\sin\beta + 2m'(R+r) \times \{\cos^{-1}[y \div (R+r)] - \beta\},$$

the equation to the required path.

Mr. Adcock finds for the equation of the curve, (in which a = Siverly's β),

$$x = \frac{2m'(r+R)}{7m+2m'}\left[\sin^{-1}\left(\frac{[2(r+R)y - y^2]^{\frac{1}{2}}}{r+R}\right) - a\right] + \left[2(r+R)y - y^2\right]^{\frac{1}{2}}.$$

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90. "Let an oblate ellipsoid of revolution of homogeneous density rotate about one of its greatest diameters." &c. (See page 160).

SOLUTION BY WALTER SIVERLY.

The attraction of the ellipsoid at any point in the shorter diameter at a distance p from the center, as shown in works on attraction, is

$$4\pi\delta k_1 p\left[\frac{1}{e^2} - \frac{\sqrt{(1 - e^2)}}{e^3}\sin^{-1}e\right] = Pp,$$

and the centrifugal force $= pa^2$. Also the attraction at any point on the longer diameter is

$$2\pi\delta k_1 p\left[\frac{\sqrt{(1 - e^2)}}{e^3}\sin^{-1}e - \frac{1 - e^2}{e^2}\right] = Qp,$$

and the centrifugal force $= pa^2$. Let u = the pressure at the point distant p from the center on the shorter diameter then

$$du = -(P - a^2)pdp. \qquad u = C - \tfrac{1}{2}p^2(P - a^2).$$

At the surface the pressure $= 0$.

$\therefore\ C = \frac{1}{2}b^2(P - a^2)$, $u = \frac{1}{2}(b^2 - p^2)(P - a^2)$, $= \frac{1}{2}b^2(P - a^2)$ at the center where $p = 0$. Similarly the pressure at the center on the longer diameter $= \frac{1}{2}a^2(Q - a^2)$. Hence for equilibrium,

$$b^2(P - a^2) = a^2(Q - a^2), \text{ or } (1 - e^2)(P - a^2) = Q - a^2.$$

Substituting values of P and Q and reducing,

$$\frac{3(1-e^2)}{e^2} - \frac{(3-2e^2)\sqrt{(1-e^2)}}{e^3}\sin^{-1}e + \frac{e^2\alpha^2}{2\pi\delta k_1} = 0.$$

When $\alpha^2 \div \delta k_1$ is known the solution of this transcendental equation gives the required ratio of the axes.

[It is well known that there are two ellipsoids of revolution that may be in equilibrium, viz., one of great, and one of small ellipticity. But this question assumes, and the solution is based on the assumption, that an ellipsoid *of revolution* may have its three principal axes unequal. If this is true then there must be an infinite number of ellipsoids of revolution in equilibrium. We have always understood an ellipsoid of revolution, however, to be a spheroid, having all its equatoral diameters equal.—Ed.]

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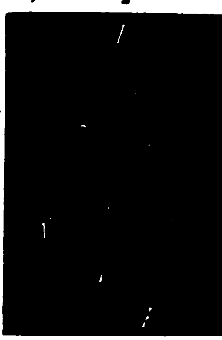
91. "Let a sphere, rotating with the angular velocity w," &c. See p. 88.

SOLUTION BY PROF. W. W. JOHNSON, ANNAPOLIS, MD.

Let A be any particle (the figure being drawn upon the spherical shell in which A is situated) C the original axis, B_1 the new axis, and B_2 the axis of the ring; then $CB_1 = \alpha$, and $B_1B_2 = 90°$. Denote the distances of A from these points as in the fig., and the linear velocities of A, due to the original rotation and the component rotations about B_1 and B_2, by V v_1 and v_2 respectively then



$$V = w\sin\beta,$$
$$v_1 = w\cos\alpha\sin\gamma,$$
$$v_2 = w\sin\alpha\sin\delta.$$

If the linear velocity V is the resultant of v_1 and v_2, there will be no sudden change in the velocity or direction of A. To show that this is the case, it is necessary to prove that the velocities are proportional to the sines of the angles taken respectively each between the directions of the other two. These directions are at right angles to δ, β and γ, and therefore make the same angles which these arcs make with one another. Now denoting the whole angle B_2AB_1 by A, and the part CAB_1 by A_1, we have by spher. trigonometry, since $\sin B_1B_2 = 1$,

$$\sin A = \frac{\sin B_1}{\sin\delta}, \quad \text{and} \quad \frac{\sin A_1}{\sin\alpha} = \frac{\sin B_1}{\sin\beta},$$

hence
$$\frac{\sin A}{\sin A_1} = \frac{\sin\beta}{\sin\alpha\sin\delta} = \frac{V}{v_2}.$$

In like manner we may prove $\sin A : \sin A_2 :: V : v_1$.

Note by the editor.—As it has been objected that, in the equation we gave, on page 143, (No. 5) of the parallel curve to an ellipse, our result was not satisfactory because the equation involved explicitly only one of the coordinates; we therefore add, as a supplement to that note, the following:

By similar triangles (see fig. on p. 143) we have

$$y - y' : x - x' :: y : ON' \quad . \quad . \quad . \quad . \quad . \quad . \quad (1')$$

Also, from the differential triangle,

$$(dx^2 + dy^2)^{\frac{1}{2}} : dy :: N : ON' \quad . \quad . \quad . \quad . \quad . \quad . \quad (2')$$

From (1′) and (2′) we get

$$N = \frac{(x - x')y}{y - y'}\sqrt{\left(1 + \frac{dx^2}{dy^2}\right)} = \frac{(x - x')y}{y - y'}\sqrt{\left(1 + \frac{a^4 - a^2x^2}{b^2x^2}\right)}.$$

Writing q for $N' \div (N' + c)$, we get by reduction,

$$x' = x - \frac{N(1 - q)bx}{\sqrt{[a^4 - (a^2 - b^2)x^2]}}.$$

Substituting for y, in the equation of the ellipse, its value from (2), (p. 143) we have $x = \frac{a}{b}\sqrt{\left(b^2 - q^2y'^2\right)}$; and putting this value of x for x, in the last equation above, and for N', as involved in q, and N, their values in functions of y' from the equation given on page 143, we obtain an equation involving only as variables the coordinates x' and y'.

PROBLEMS.

92. By A. W. Mason, Cedar Falls, Iowa.—A balloon is ascending vertically with a given velocity v, and a body is let fall from it, which touches the ground in t seconds; find the height of the balloon at the moment the body is let fall fron it.

93. By Prof. J. Scheffer.—To construct a triangle if the three radii of the circles, which touch the three sides externally, be given.

94. By Prof. W. W. Johnson.—One side of a quadrilateral, whose four sides are given in length, is fixed: find the equation of the locus of the middle point of the opposite side, in rectangular coordinates.

95. By Dr. A. B. Nelson, Danville, Ky.—Prove, otherwise than by the Integral Calculus that

$$\frac{\pi}{2} - \sin^{-1}e = 2\tan^{-1}\left(\frac{1 - e}{1 + e}\right)^{\frac{1}{2}}.$$

96. By Chas. P. Saxe, Gold Hill, Nevada. — A plays m games with B whose skill is equal to his own. Required the probability that one of them will win n consecutive games.

97. By Henry Gunder, North Manchester, Ind.—Find the average distance of all the points of a sphere, radius r, from a point whose distance from the center is a.

Query. By G. W. Hill.—In De Haan's Tables of Definite Integrals, Edition of 1867, p. 317, there is found this equation

$$\int_0^{\frac{\pi}{2}} \frac{x \sin x}{(1 - p^2 \sin^2 x)^{\frac{3}{2}}}\, dx = \frac{\sin^{-1} p}{p(1 - p^2)},$$

how prove this?

Editorial Announcement. — As this number completes Vol. II, we take this opportunity to announce that we have made definite arrangements for the continuation of the Analyst at least another year, and we hope, for several years.

It is not our purpose to boast of the character of our publication, as we are fully aware of its many defects, but we are pleased to know that many of the best mathematicians in the country give us their active support, both by way of contributions for publication, and pecuniary aid: As, for instance, Mr. Hill, who has enriched our pages with many valuable articles and solutions, has contributed to this No. a very interesting, and some what extended, article; and, as it occupies more space than is in general allowed to any one contributor in the same No., he has generously allowed us to add 8 pages to the No. at his expense; thus enabling us to present to our readers a No. of 40 pages, without increase of cost to them or expense to us.

We have thought several times during the past two years that an apology was due for the quality of the paper used in some of the numbers. We dislike apologies, however, and will only state that we *paid* for good paper. We take pleasure in saying, in this connection, that, by the kindness of the Secretary of State, we have been permitted to include our purchase of paper for Vol. III, with the purchase made for the state, whereby we have obtained, at a reduced price, a superior article of paper; as will appear by inspection of the last two sheets of this No.

We desire to retain *all* our present subscribers, and would be pleased if each of them would procure for us an *additional* one or more, but, in case any of our present subscribers intend to discontinue their subscriptions at the end of Vol. II, we ask as a favor, as well as an act of justice to us, that they notify us *before the first of December*, 1875.